Imprecise Probability Theory

Basic Concepts and Credal Sets

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SIPTA School on Imprecise Probability

Pescara, July 16th, 2012

Outline and scheduling

- An informal introduction to IPs
    - Reasoning with (imprecise) fault trees
    - From determinism to imprecision (through uncertainty)
- What probability is? (aka does probability exists?)
    - Subjective vs. Objective
    - Behavioural interpretation of subjective probability
    - Prices, probabilities, previsions
- What imprecise probability is?
    - Reasons for imprecise probabilities
    - Avoiding sure loss, coherence and natural extension
    - Credal sets

Reasoning: from Determinism to IPs

brake fails = [pads $\vee$ (sensor $\wedge$ controller $\wedge$ actuator)]

devices failures are independent

Reasoning: from Determinism to IPs

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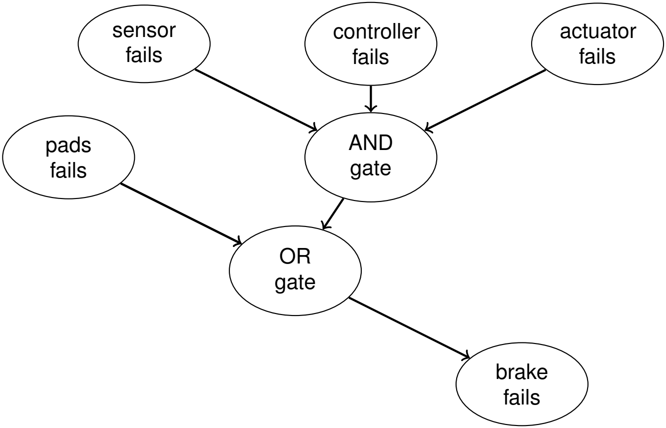

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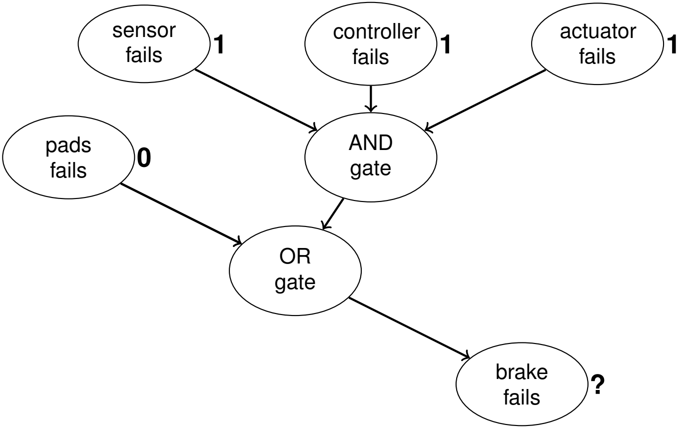

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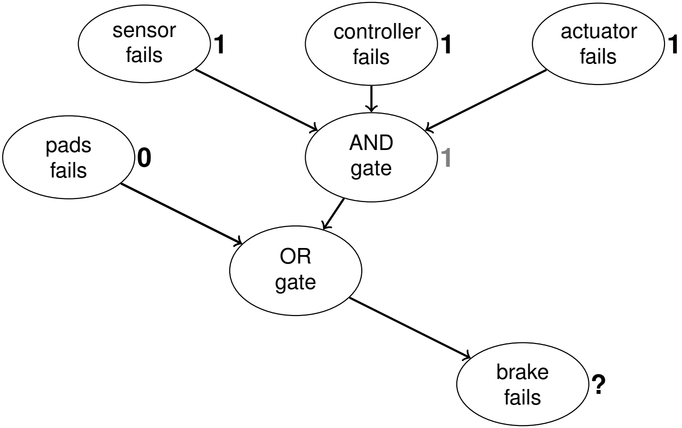

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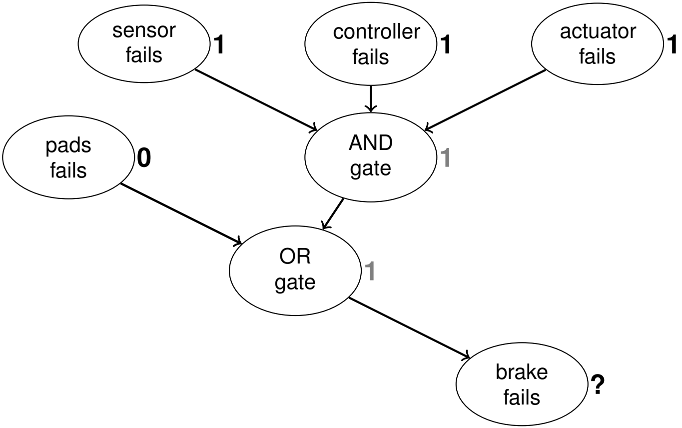

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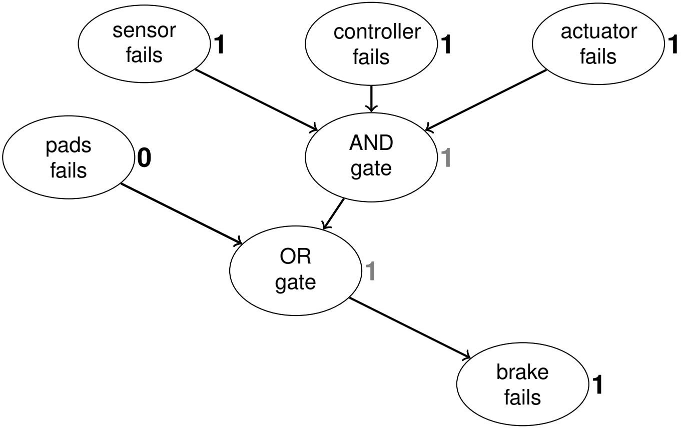

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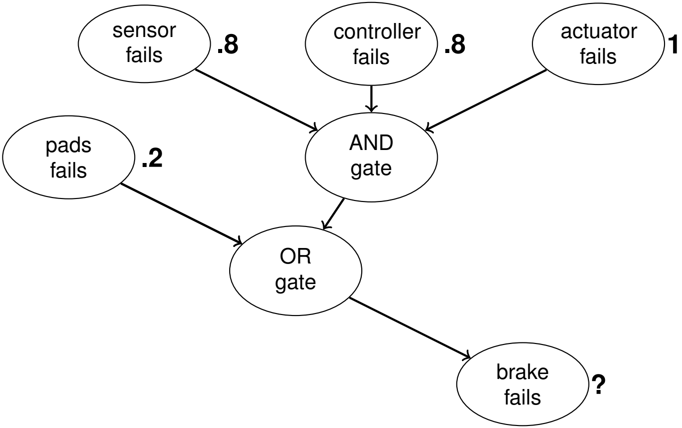

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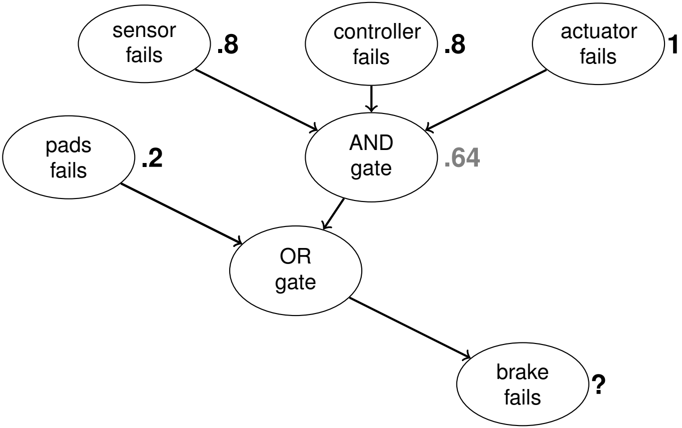

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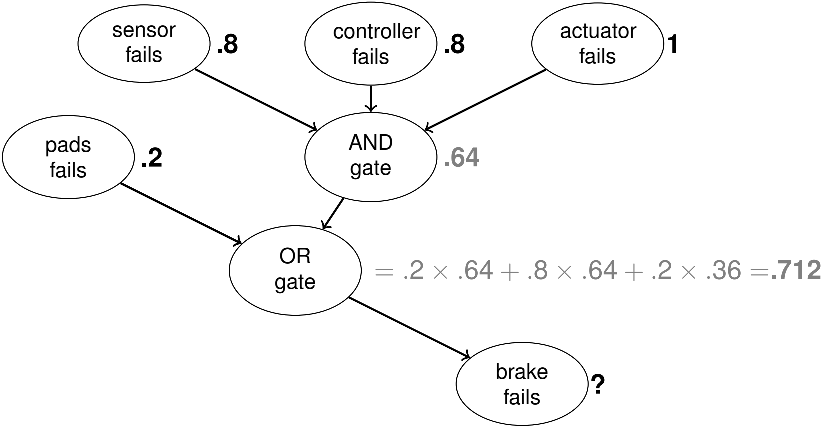

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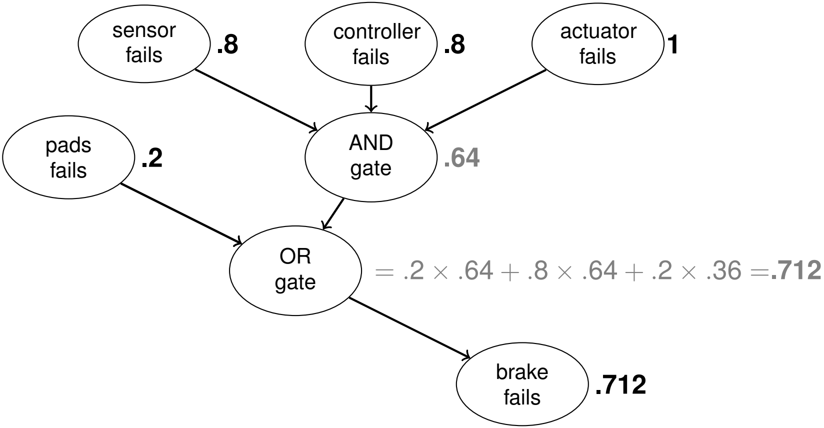

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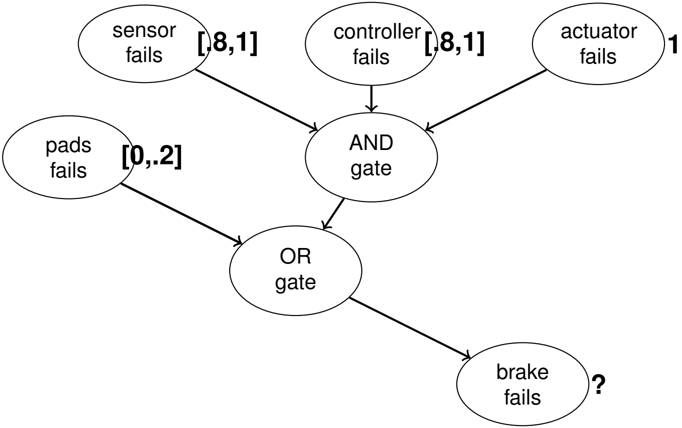

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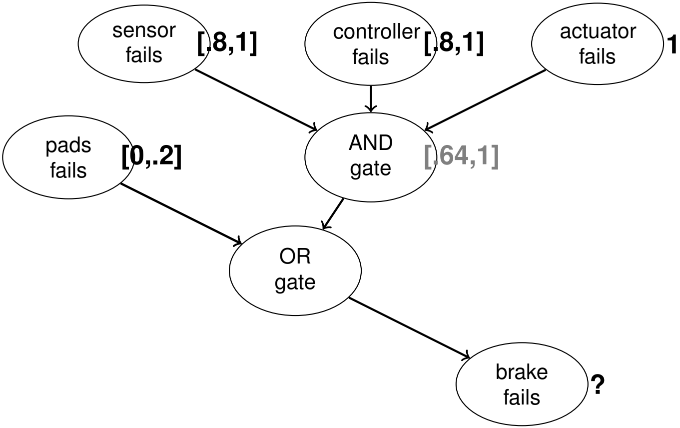

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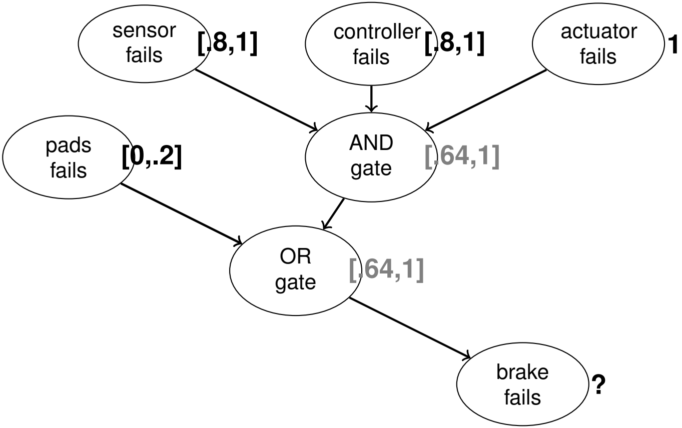

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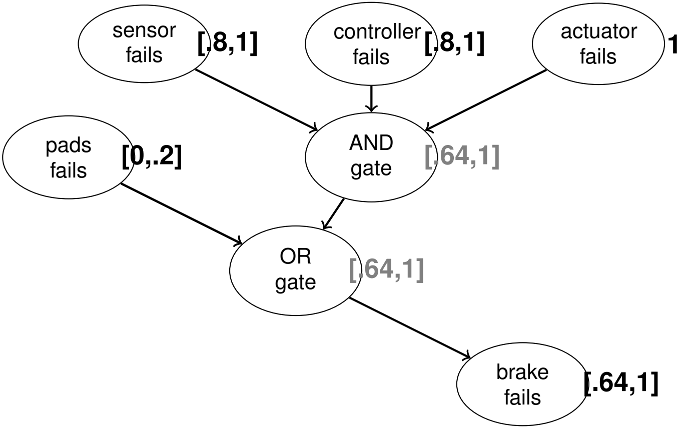

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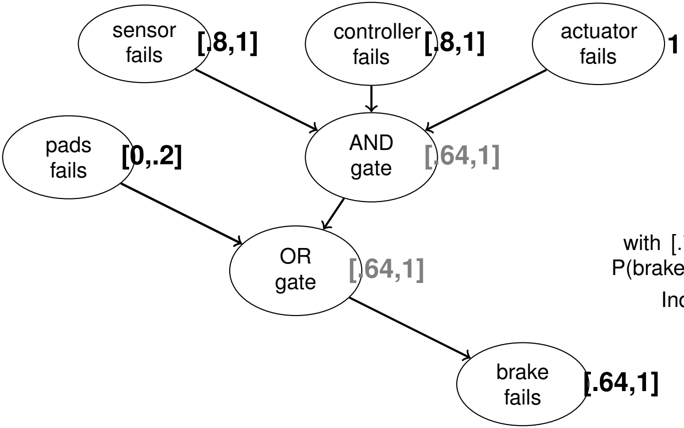

with [.7, 1] instead
P(brake fails)∈[.49,1]

Indecision!

devices failures are independent

Three different levels of knowledge

- FIFA’12 final match between Italy and Spain
- Result of Spain after the regular time? Win, draw or loss?

DETERMINISM

The Spanish goalkeeper is unbeatable and Italy always receives a goal

Spain (certainly) wins

$$\begin{matrix} P(Win) \\ P(Draw) \\ P(Loss) \end{matrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

UNCERTAINTY

Win is two times more probable than draw, and this being three times more probable than loss

$$\begin{matrix} P(Win) \\ P(Draw) \\ P(Loss) \end{matrix} = \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$$

IMPRECISION

Win is more probable than draw, and this is more probable than loss

$P(Win) > P(Draw)$
$P(Draw) > P(Loss)$

$$\begin{matrix} P(Win) \\ P(Draw) \\ P(Loss) \end{matrix} = \begin{bmatrix} \frac{\alpha}{3} + \beta + \frac{\gamma}{2} \\ \frac{\alpha}{3} + \frac{\gamma}{2} \\ \frac{\alpha}{3} \end{bmatrix}$$

$\forall \alpha, \beta, \gamma$ such that
$\alpha > 0, \beta > 0, \gamma > 0,$
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Three different levels of knowledge

DETERMINISM UNCERTAINTY IMPRECISION

Three different levels of knowledge

DETERMINISM UNCERTAINTY IMPRECISION

INFORMATIVENESS

Three different levels of knowledge

DETERMINISM UNCERTAINTY IMPRECISION

EXPRESSIVENESS

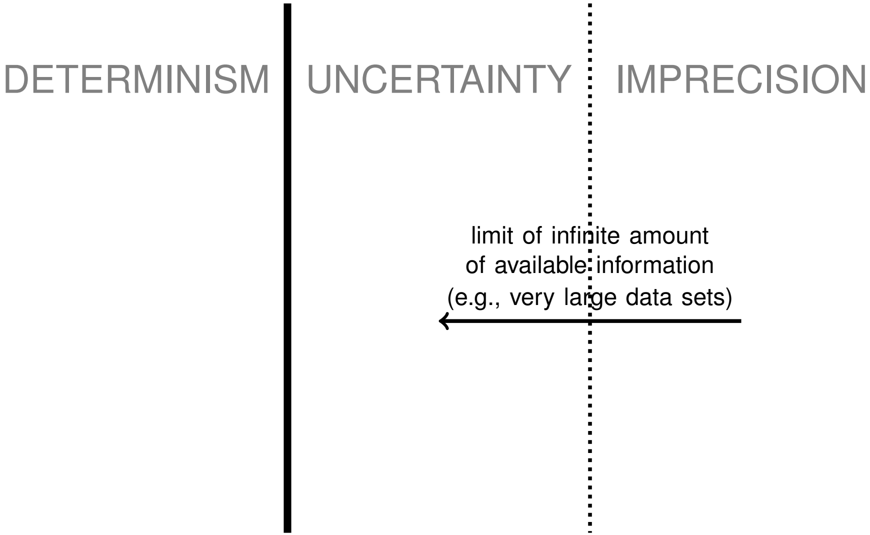
Three different levels of knowledge

DETERMINISM

UNCERTAINTY

IMPRECISION

Three different levels of knowledge



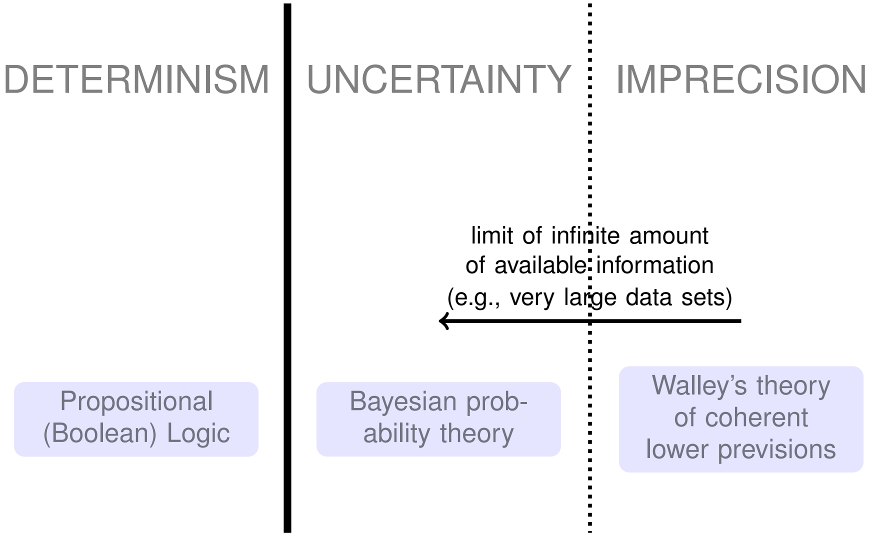
DETERMINISM

UNCERTAINTY

IMPRECISION

limit of infinite amount
of available information
(e.g., very large data sets)

Three different levels of knowledge



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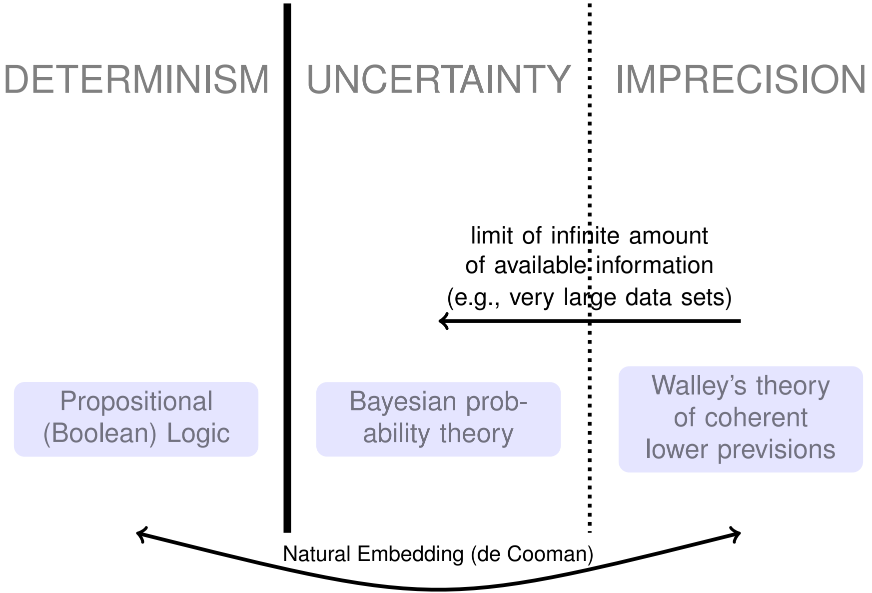
limit of infinite amount of available information (e.g., very large data sets)

Propositional (Boolean) Logic

Bayesian probability theory

Walley's theory of coherent lower previsions

Three different levels of knowledge



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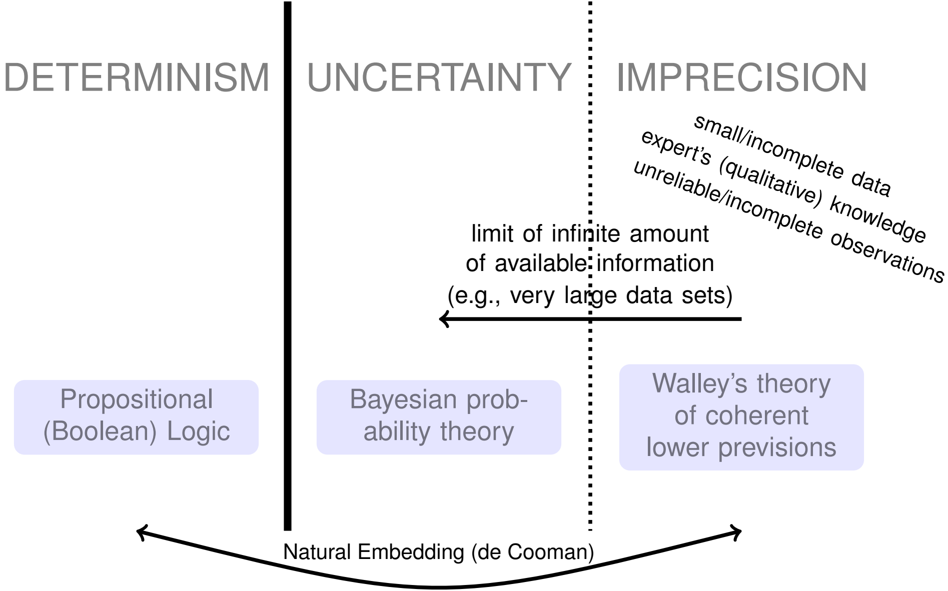
Natural Embedding (de Cooman)

Three different levels of knowledge



DETERMINISM

UNCERTAINTY

IMPRECISION

small/incomplete data
expert's (qualitative) knowledge
unreliable/incomplete observations

limit of infinite amount
of available information
(e.g., very large data sets)

Propositional (Boolean) Logic

Bayesian probability theory

Walley's theory of coherent lower previsions

Natural Embedding (de Cooman)

[. . .] Bayesian inference will always be a basic tool for practical everyday statistics, if only because questions must be answered and decisions must be taken, so that a statistician must always stand ready to upgrade his vaguer forms of belief into precisely additive probabilities

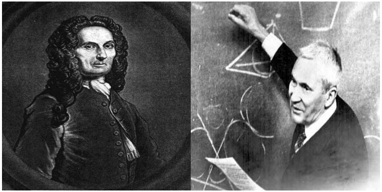
Art Dempster (in his foreword to Shafer's book)

Probability: one word for two (not exclusive) things

Randomness

Variability captured through repeated observations

De Moivre and Kolmogorov



- Chances
- Feature of the world
- Aleatory or objective
- Frequentist theory
- Limiting frequencies

Partial knowledge

Incomplete information about issues of interest

Bayes and De Finetti

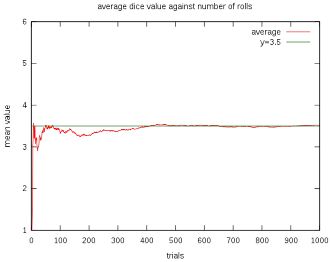
- Beliefs
- Feature of the observer
- Epistemic or subjective
- Bayesian theory
- Behaviour (bets dispositions)

Objective probability

- X taking its values in (finite set) Ω
- Value $X = x \in \Omega$ as the output of an experiment which can be iterated
- Prob $P(x)$ as limiting frequency

$$P(x) := \lim_{N \to +\infty} \frac{\#(X = x)}{N}$$

- Kolmogorov's axioms follow from this
- Probability as a property of the world
- Not only (statistical and quantum) mechanics, hazard games (coins, dices, cards), but also economics, bio/psycho/sociology, linguistics, etc.
- But not all events can be iterated . . .



1. $\forall A \in 2^{\Omega}, 0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $\forall A, B \in 2^{\Omega} : A \wedge B = \emptyset$ $P(A \vee B) = P(A) + P(B)$

Probability in everyday life

sky TG24 IARONI: PUGNO DURO SE LE VIOLENZE NON CESSERANNO - SCALATA UNIPOL A BI

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IN EVIDENZA

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Afghanistan 2.0
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Calabria
Calendari
Campania
Casa
Clienti
Cronaca
Diretta Skytg24
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Guido Bertolaso
Haiti
Indonesia
Io Reporter
Iraq
Libri
Lodo Alfano
Maltempo

tg24 > politica

Bersani: Il voto a primavera? "Ci sono buone probabilità"

Il segretario del Pd rilancia il progetto del Nuovo Ulivo e offre una proposta di alleanza di governo rivolta al cosiddetto Terzo Polo. Poi sdrammatizza con una battuta le voci sul futuro leader del partito e rilancia le primarie: "Una cosa bellissima"

10 ottobre, 2010

Il segretario del Pd Pier Luigi Bersani

Guarda la Fotogallery

hoy.com.ec
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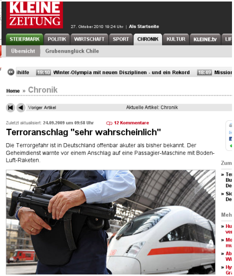
Chile: probabilidades de encontrar mineros con vida 'son bajas'

Publicado el 12/Agosto/2010 | 09:10

SANTIAGO. Las probabilidades de encontrar con vida a los 33 mineros atrapados desde hace una semana en un mina en el norte de Chile "son bajas", admitió el ministro de Minería, Laurance Golborne, que encabeza las labores de búsqueda.

"Las probabilidades de encontrarlos con vida son bajas", señaló el ministro en una entrevista con Canal 13, siete días después que un derrumbe sepultó a 33 mineros en el yacimiento San José, en las cercanías de Copiapó, a unos 800 km al norte de Santiago.



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STEIERMARK POLITIK WIRTSCHAFT SPORT CHRONIK KULTUR KLEINE.tv

Übersicht Grubenunglück Chile

ihilfe 19:10 Winter-Olympia mit neuen Disziplinen - und ein Rekord 18:49 Mission

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Voriger Artikel Aktuelle Artikel: Chronik

Zuletzt aktualisiert: 24.09.2009 um 09:58 Uhr 12 Kommentare

Terroranschlag "sehr wahrscheinlich"

Die Terrorgefahr ist in Deutschland offenbar akuter als bisher bekannt. Der Geheimdienst warnte vor einem Anschlag auf eine Passagier-Maschine mit Boden-Luft-Raketen.

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DÉPÊCHE PRÉCÉDENTE DÉPÊCHE SUIVANTE

Xi Jinping, très probable successeur de Hu Jintao et "fils de prince"

AFP | 18.10.10 | 10h36

PUBLICITE

Xi Jinping, 57 ans, qui vient d'être nommé vice-président de la puissante Commission militaire

Charleston Daily Mail dailymail.com
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SPORTS

Wednesday October 20, 2010

Anderson and Sullivan both will 'probably' play for Herd

by Chuck McGill
Daily Mail sports writer

HUNTINGTON, W.Va. -- Regarding the quarterback

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Fecombustíveis alerta para provável aumento de gasolina

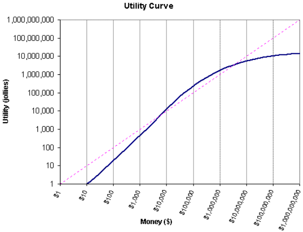
Probabilities often pertains to singular events

not necessarily related to statistics

Subjective probability

- Probability p of me having a kid
- Singular event: frequency unavailable
- Subjective probability
  - models (partial) knowledge of a subject
  - feature of the subject not of the world
  - two subjects can assess different probs
- Quantitative measure of knowledge?
  - Behavioural approach
  - Subjective betting dispositions
  - A (linear) utility function is needed

- Money?
- Big money not linear!
- Small, somehow yes



lottery tickets

$\propto$

winning chance

$\propto$

benefit

infinite number of tickets
makes utility real-valued

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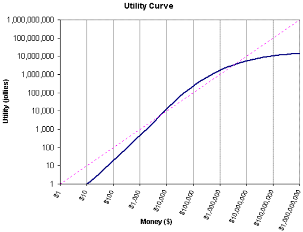

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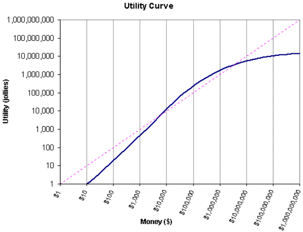

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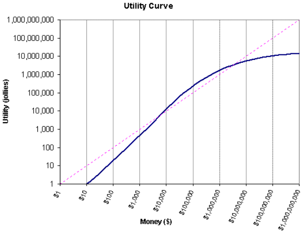

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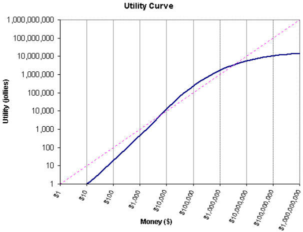

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(Rationally) betting on gambles

- Probabilities as dispositions to buy/sell gambles
- Gambles (Anglo-Saxon world) are checks whose amount is uncertain/unknown

This check has a value of 100 EUR if Alessandro has a child zero otherwise

- The bookie sells this gamble
- Probability *p* as a *price* for the gamble
  - $\frac{\text{maximum price}}{\text{100EUR}}$ for which you buy the gamble
  - $\frac{\text{minimum price}}{\text{100EUR}}$ for which you (bookie) sell it
- Interpretation + rationality produce axioms

(Rationally) betting on gambles

- Probabilities as dispositions to buy/sell gambles
- Gambles (Anglo-Saxon world) are checks whose amount is uncertain/unknown

This check has a value of 100 EUR if Alessandro has a child zero otherwise

- The bookie sells this gamble
- Probability *p* as a *price* for the gamble
  - $\frac{\text{maximum price}}{\text{100EUR}}$ for which you buy the gamble
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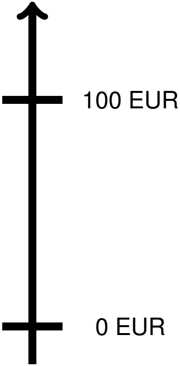

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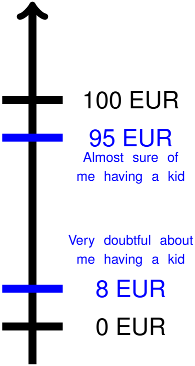

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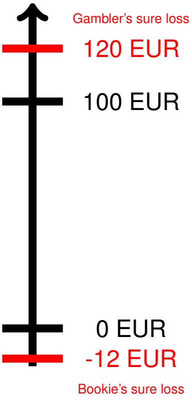

Gambler's sure loss

120 EUR

100 EUR

0 EUR

-12 EUR

Bookie's sure loss

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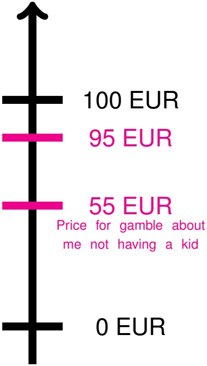

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100 EUR

95 EUR

55 EUR

Price for gamble about me not having a kid

0 EUR

You spend 150 EUR to (certainly) win 100 EUR!

Coherence and linear previsions

Don't be crazy: choose prices s.t.
there is always has a chance to win
(whatever the stakes set by the bookie)

Prices $\{P_{A_i}\}_{i=1}^{N}$ for events $A_i \subseteq \Omega$, $i = 1, \ldots, N$ are coherent iff

$$\max_{x \in \Omega} \sum_{i=1}^{N} c_i [I_{A_i}(x) - P_{A_i}] \geq 0$$

Moreover, assessments $\{P_{A_i}\}_{i=1}^{N}$ are coherent iff

- Exists probability mass function $P(X)$: $P(A_i) = P_{A_i}$
- Or, for general gambles, linear functional $P(f_i) := P_{f_i}$

$$P(f) = \sum_{x \in \Omega} P(x)\, f(x)$$

linear prevision
to be extended to a
coherent lower prevision

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to be extended
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Subjective P, if objective P exists?

- Chances known $\Rightarrow$ beliefs coincide
- Swiss lotto (45 nums): X_6 next lotto's 6-tuple
- x'_6 and x''_6 your two guesses (6 + 6 nums)
- $P((X_6 = x'_6) \vee (X_6 = x''_6)) \simeq 1/4,000,000$
  - You spend $\frac{1}{4,000,000}$ to have 1 CHF if you win
  - You spend $\frac{s}{4,000,000}$ to have s CHF if you win
  - But $\frac{s}{4,000,000} = 3$ CHFs $\Rightarrow s = 12,000,000$
  - Worth play if jackpot $\geq$ 12'000'000 CHFs

Joker 6 4 3 7 5 3

TT.MM. ANZEIGEN »

4 5 15 17 30 32 33

3

3

Ziehung vom 27.10.2010	Gewinnränge		Anzahl Gewinner	CHF
	6	Plus	0	0.00 *
	6		1	7'432'979.65
	5 + ZZ	Plus	2	82'043.80 *
	5 + ZZ		2	35'786.20
	5	Plus	3	15'572.25 *
Jackpot	5		57	3'781.10
CHF 800'000	4	Plus	507	100.00 *
	4		2'897	50.00
Jackpot mit Plus	3	Plus	9'321	12.00 *
CHF 4'100'000 *	3		47'617	6.00
JETZT SPIELEN »	2	Plus	58'899	2.00

* kumulierte Gewinnquoten

6 4 3 7 5 3

Post Scriptum

My personal

$P(X = \text{true})$

is one

which means

I have a kid

(and I know that)

BREAK

(subjective, behavioural) imprecise probabilities

De Finetti's precision dogma

$$\overline{P}(x) \qquad \underline{P}(x)$$

minimum selling price $=$ maximum buying price

Walley's proposal for imprecision

No strong reasons for that rationality only requires $\underline{P}(x) \leq \overline{P}(x)$

- Avoid sure loss! With max buying prices $\underline{P}(A)$ and $\underline{P}(A^c)$, you can buy both gambles and earn one for sure:

$$\underline{P}(A) + \underline{P}(A^c) \leq 1$$

- Be coherent! When buying both A and B, you pay $\underline{P}(A) + \underline{P}(B)$ and you have a gamble which gives one if $A \cup B$ occurs:

$$\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B)$$

coherence self-consistency (beliefs revised if unsatisfied) less problematic than a.s.l.

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Reasons for imprecise probabilities

- Reflect the amount of information on which probs are based
- Uniform probs model indifference not ignorance
- When doing introspection, sometimes indecision/indeterminacy
- Easier to assess (e.g., qualitative knowledge, combining beliefs)
  Assessing precise probs could be possible in principle, but not in practice because of our bounded rationality
- Natural extension of precise models defined on some events determine only imprecise probabilities for events outside
- Robustness in statistics (multiple priors/likelihoods) and decision problems (multiple prob distributions/utilities)

Credal sets (Levi, 1980) as IP models

- Without the precision dogma, incomplete knowledge described by (credal) sets of probability mass functions
- Induced by a finite number of assessments (l/u gambles prices) which are linear constraints on the consistent probabilities
- Sets of consistent (precise) probability mass functions convex with a finite number of extremes (if $|\Omega| < +\infty$)
- E.g., no constraints ⇒ vacuous credal set (model of ignorance)

$$K(X) = \left\{ P(X) \left| \begin{array}{l} \sum_{x\in\Omega} P(x) = 1 \\ P(x) \geq 0 \end{array} \right. \right\}$$

Natural extension

- Price assessments are linear constraints on probabilities (e.g., $\underline{P}(f) = .21$ means $\sum_x P(x)f(x) \geq .21$)
- Compute the extremes $\{P_j(X)\}_{j=1}^{v}$ of the feasible region
- The credal set $K(X)$ is $\mathrm{ConvHull}\{P_j(X)\}_{j=1}^{v}$
- Lower prices/expectations of any gamble/function of/on X

$$\underline{P}(h) = \min_{P(X) \in K(X)} \sum_{x \in \mathcal{X}} P(x) \cdot h(x)$$

LP task: optimum on the extremes of $K(X)$

Computing expectations on credal sets

- Constrained optimization problem, or
- Combinatorial optimization on the extremes space

  (# of extremes can be exponential in # of constraints)

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Lower-upper conjugacy

E.g., with events

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For gambles, similarly,

$$\overline{P}(-f) = -\underline{P}(f)$$

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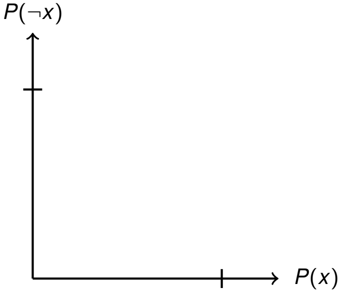

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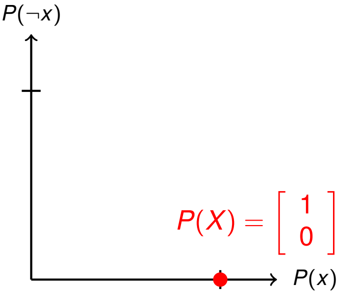

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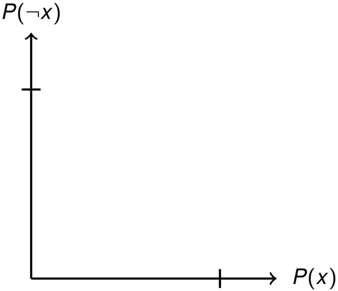

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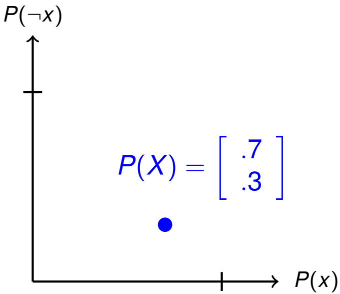

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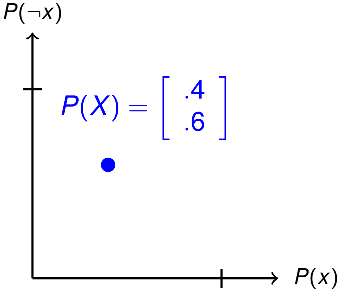

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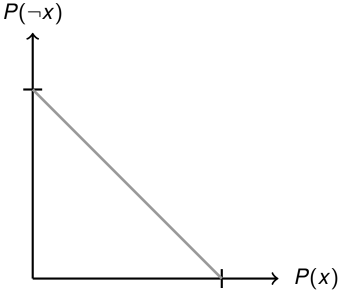

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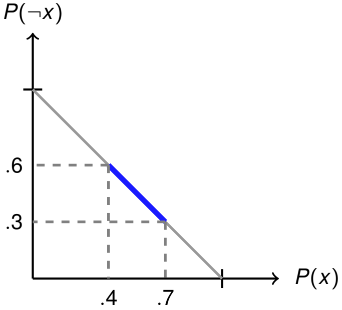
$$K(X) \equiv \left\{ P(X) = \begin{bmatrix} p \\ 1-p \end{bmatrix} \middle| .4 \leq p \leq .7 \right\}$$



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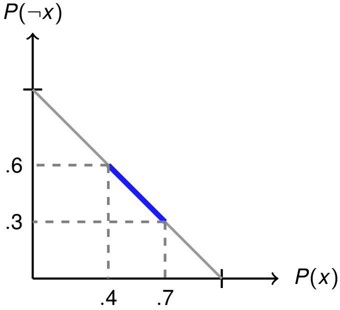
  $$K(X) \equiv \left\{ P(X) = \left[\begin{array}{c} p \\ 1-p \end{array} \right] \middle| .4 \leq p \leq .7 \right\}$$
- A CS over a Boolean variable cannot have more than two vertices!

  $$\mathrm{ext}[K(X)] = \left\{ \left[\begin{array}{c} .7 \\ .3 \end{array} \right], \left[\begin{array}{c} .4 \\ .6 \end{array} \right] \right\}$$



Geometric Representation of CSs (ternary variables)

- Ternary X (e.g., $\Omega = \{\text{win,draw,loss}\}$)
- $P(X) \equiv$ point in the space (simplex)
- No bounds to $|\text{ext}[K(X)]|$
- Modelling ignorance
  - Uniform models indifference
  - Vacuous credal set
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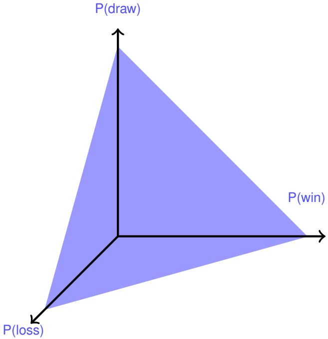

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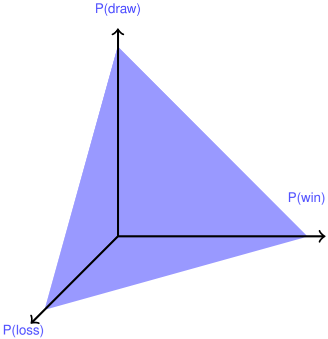

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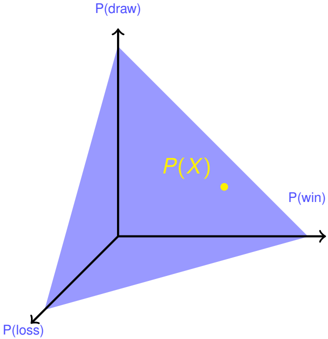

$$P(X) = \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$$

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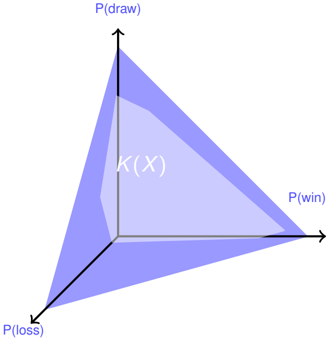

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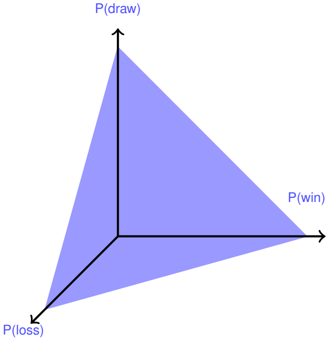

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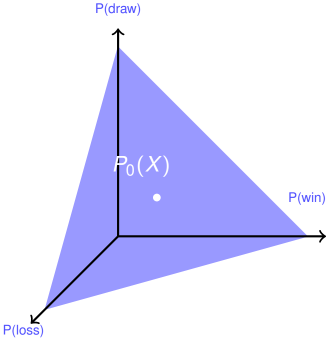

$$P_0(x) = \frac{1}{|\Omega_X|}$$

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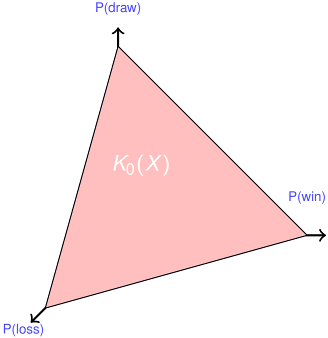

$$K_0(X) = \left\{ P(X) \;\middle|\; \begin{array}{l} \sum_x P(x) = 1, \\ P(x) \geq 0 \end{array} \right\}$$

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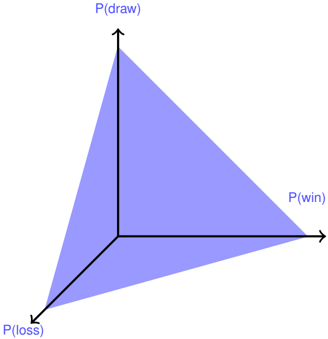

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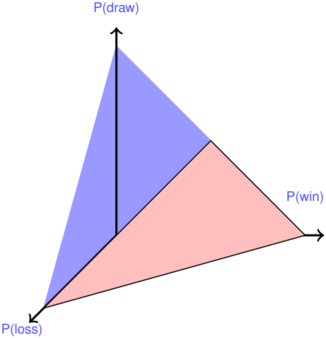

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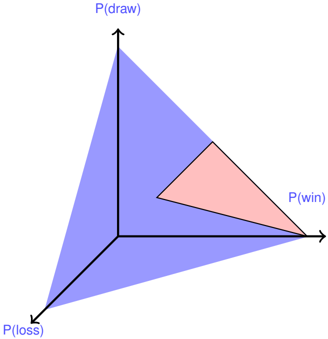

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From natural language to linear constraints on probabilities

(Walley, 1991)

extremely probable $P(x) \geq 0.98$
very high probability $P(x) \geq 0.9$
highly probable $P(x) \geq 0.85$
very probable $P(x) \geq 0.75$
has a very good chance $P(x) \geq 0.65$
quite probable $P(x) \geq 0.6$
probable $P(x) \geq 0.5$
has a good chance $0.4 \leq P(x) \leq 0.85$
is improbable (unlikely) $P(x) \leq 0.5$
is somewhat unlikely $P(x) \leq 0.4$
is very unlikely $P(x) \leq 0.25$
has little chance $P(x) \leq 0.2$
is highly improbable $P(x) \leq 0.15$
is has very low probability $P(x) \leq 0.1$
is extremely unlikely $P(x) \leq 0.02$

Marginalization (and credal sets in 4D)

- Two Boolean variables: **S**moker, Lung **C**ancer
- 8 “Bayesian” physicians, each assessing $P_j(S,C)$ $K(S,C) = \mathrm{CH}\{P_j(S,C)\}_{j=1}^{8}$

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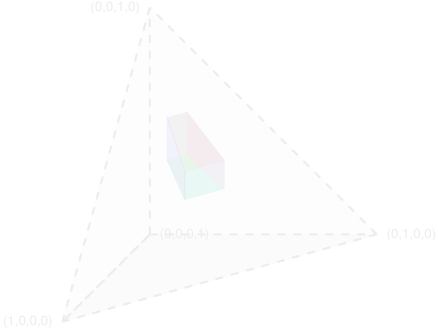
Marginals elementwise (on extremes)

$$K(C) = CH\left\{\sum_s P_j(C,s)\right\}_{j=1}^{8} \quad \frac{1}{2} \leq P(c) \leq \frac{3}{4}$$



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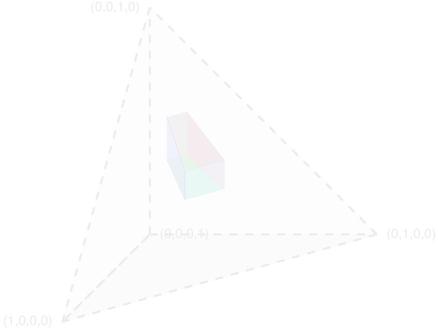

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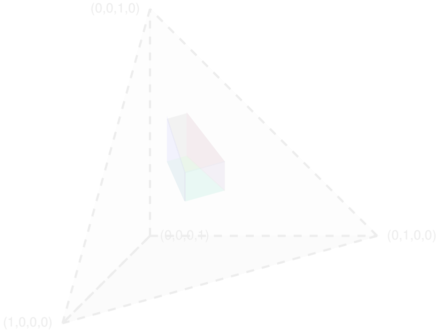

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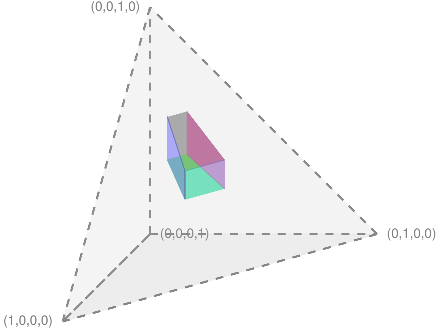

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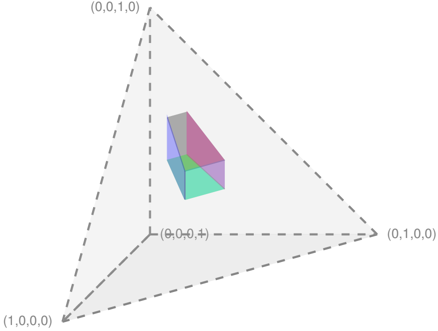
8	3/8	1/8	3/8	1/8

Marginals elementwise (on extremes)

$$K(C) = CH\left\{\sum_s P_j(C,s)\right\}_{j=1}^{8} \quad \frac{1}{2} \leq P(c) \leq \frac{3}{4}$$



Credal sets induced by probability intervals

- Assessing lower and upper probabilities: $[l_x, u_x]$, for each $x \in \Omega$
- The consistent credal set is

$$K(X) := \left\{ P(X) \left| \begin{array}{l} l_x \leq P(x) \leq u_x \\ P(x) \geq 0 \\ \sum_x P(x) = 1 \end{array} \right. \right\}$$

- Avoiding sure loss implies non-emptiness of the credal set

$$\sum_x l_x \leq 1 \leq \sum_x u_x$$

- Coherence implies the reachability (bounds are tight)

$$u_x + \sum_{x' \neq x} l_x \leq 1 \qquad l_x + \sum_{x' \neq x} u_x \geq 1$$

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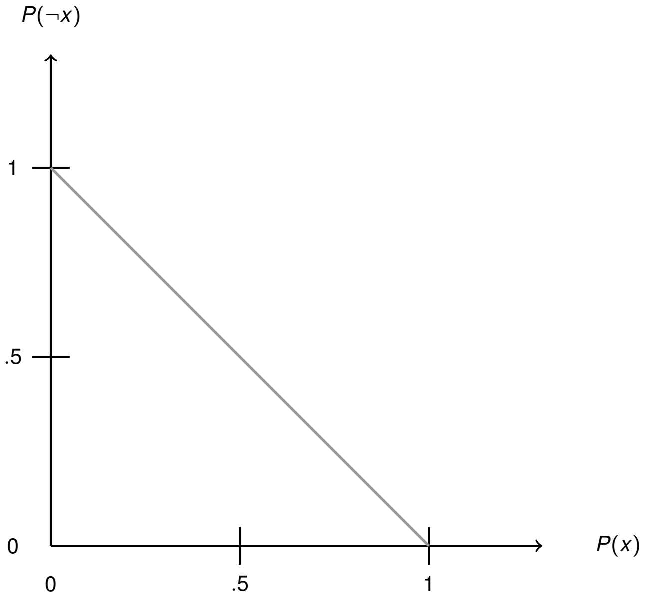
Refining assessments (when possible)



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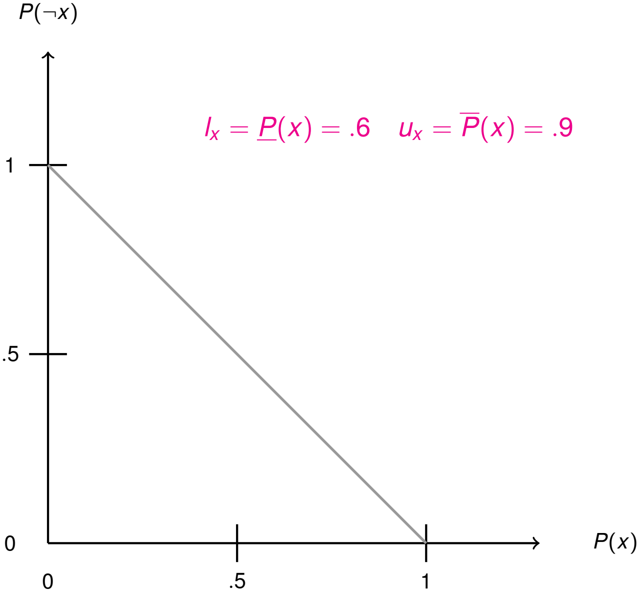

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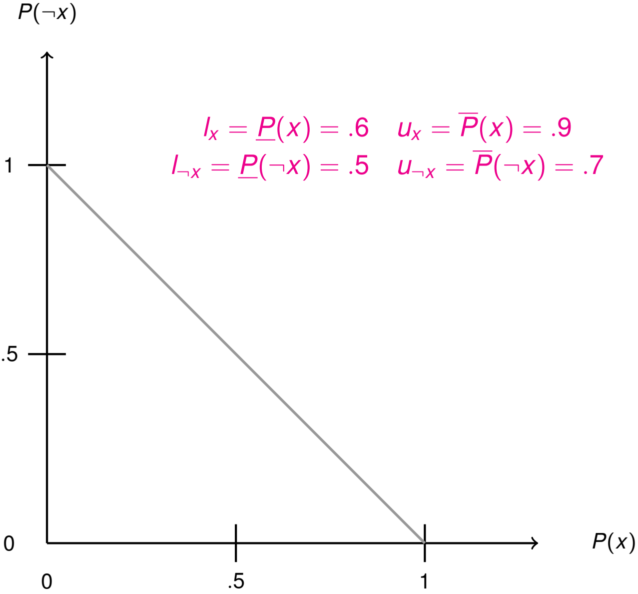

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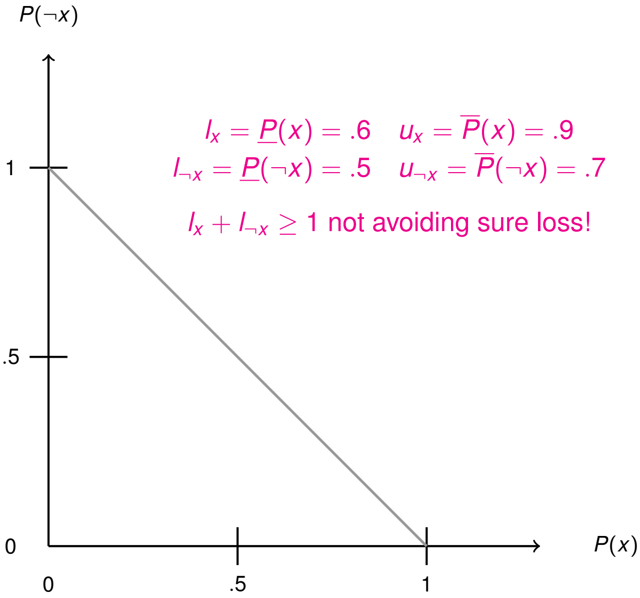

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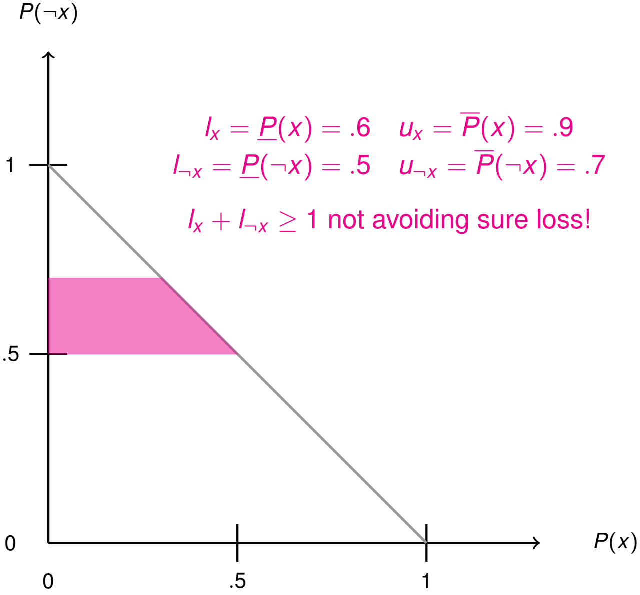

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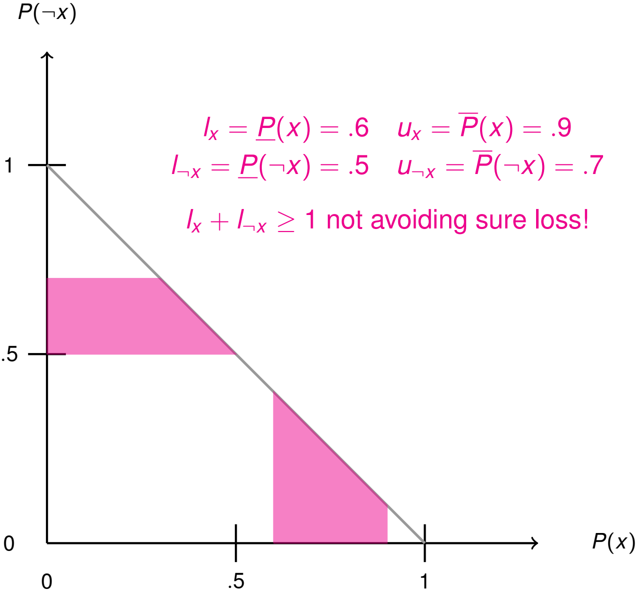

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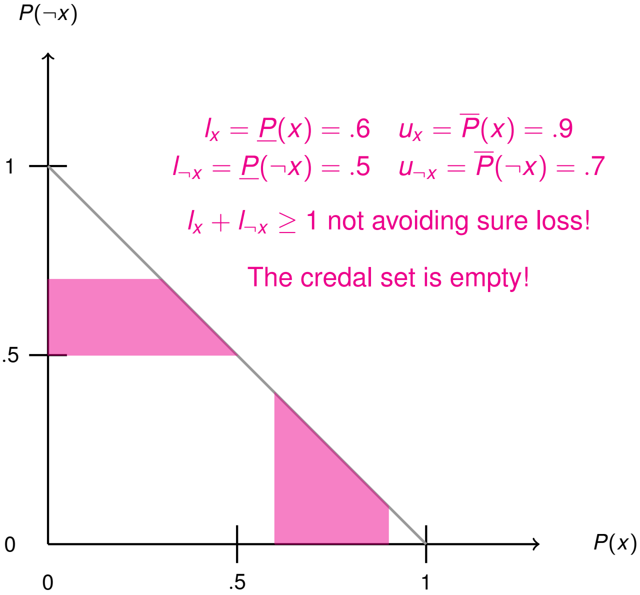

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The credal set is empty!

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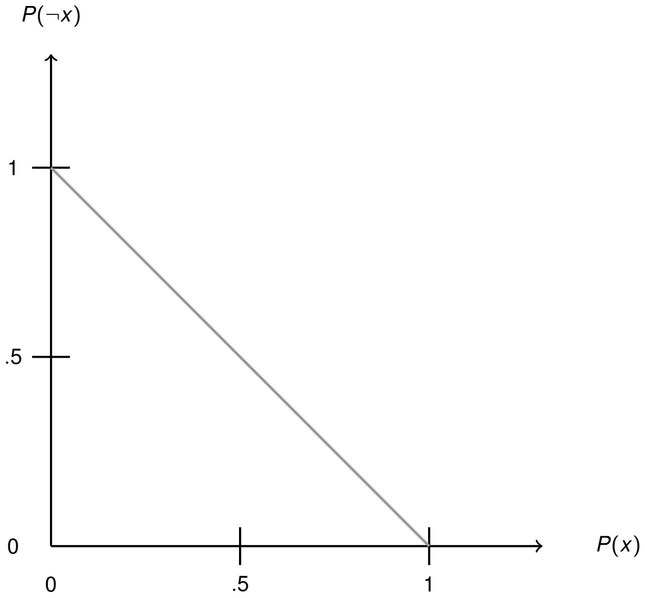

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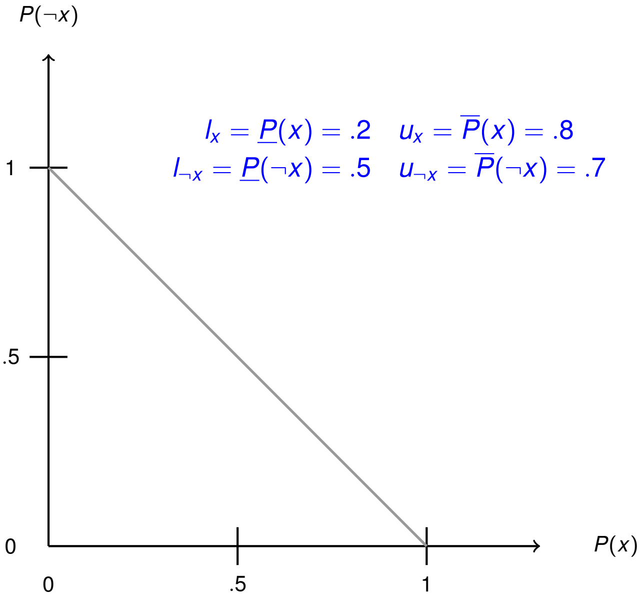

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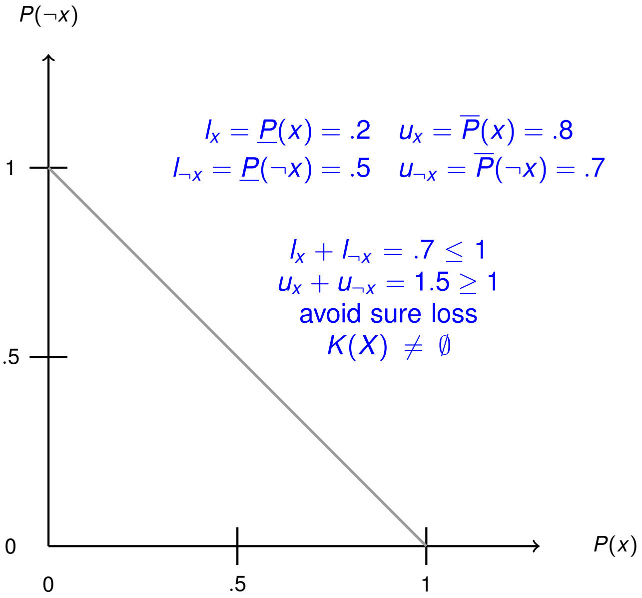

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avoid sure loss

$K(X) \neq \emptyset$

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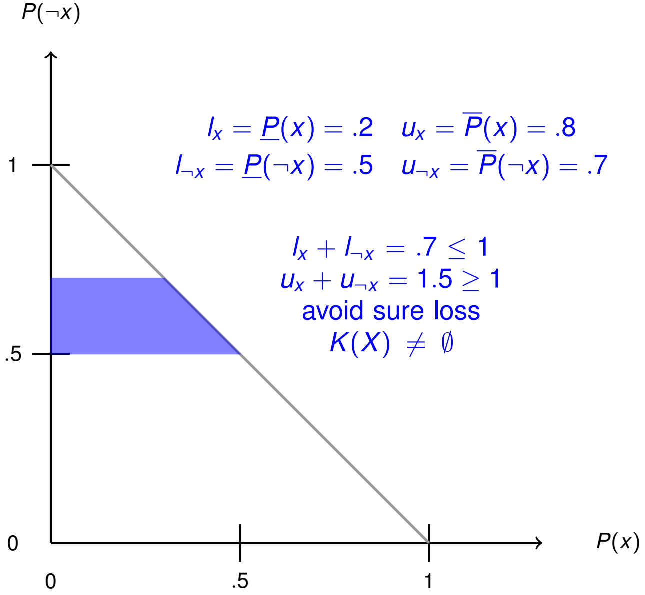

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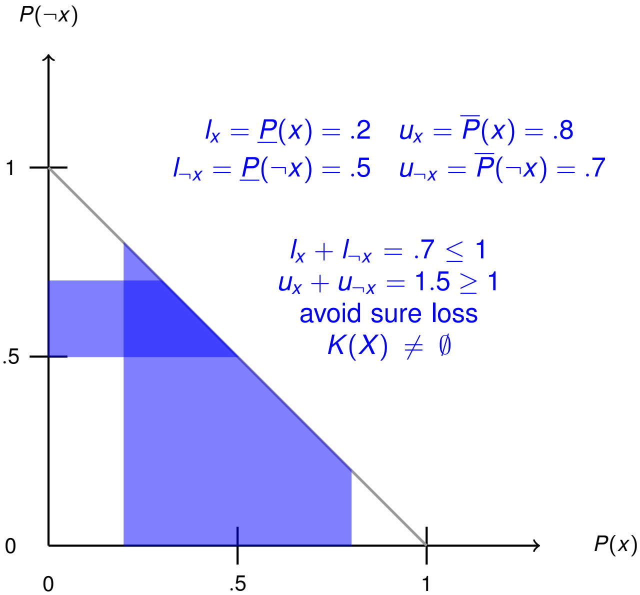

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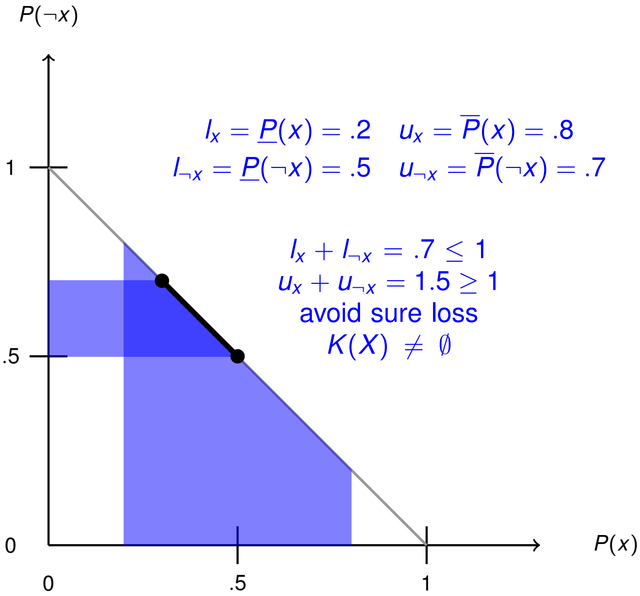

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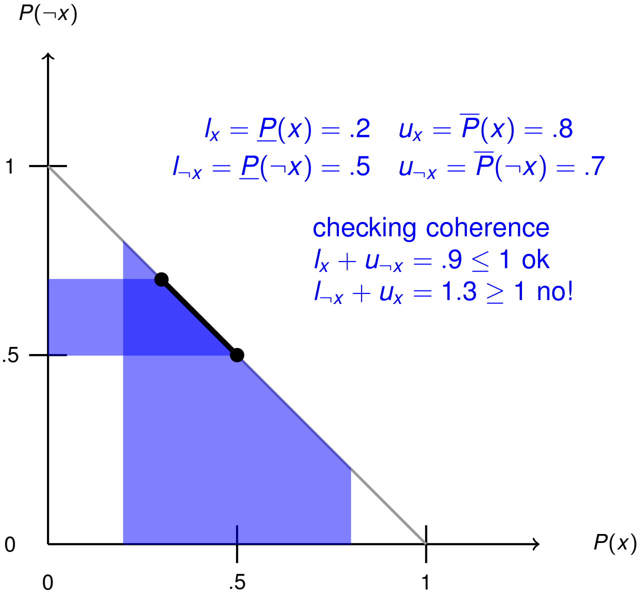

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$l_x + u_{\neg x} = .9 \leq 1$ ok

$l_{\neg x} + u_x = 1.3 \geq 1$ no!

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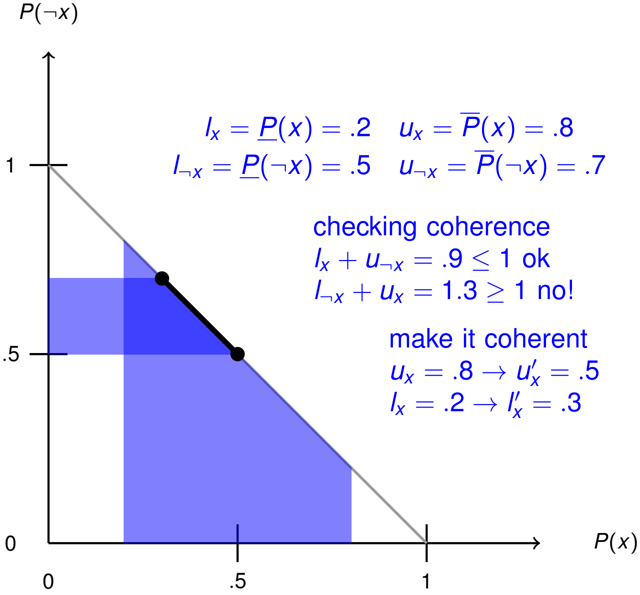

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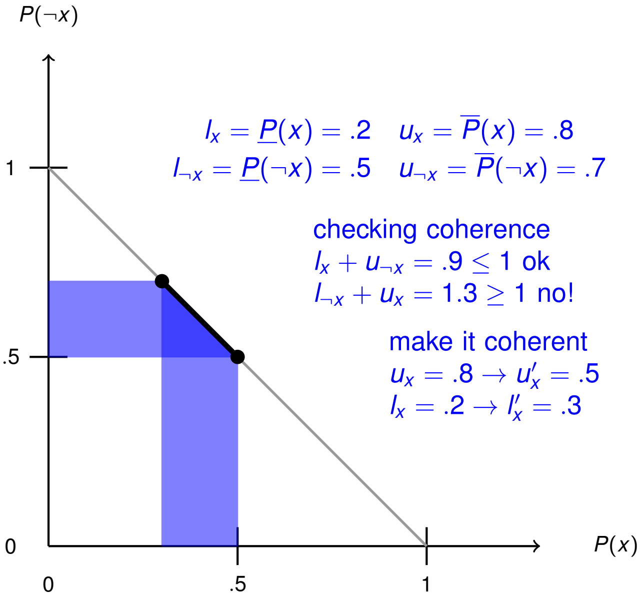
Refining assessments (when possible)



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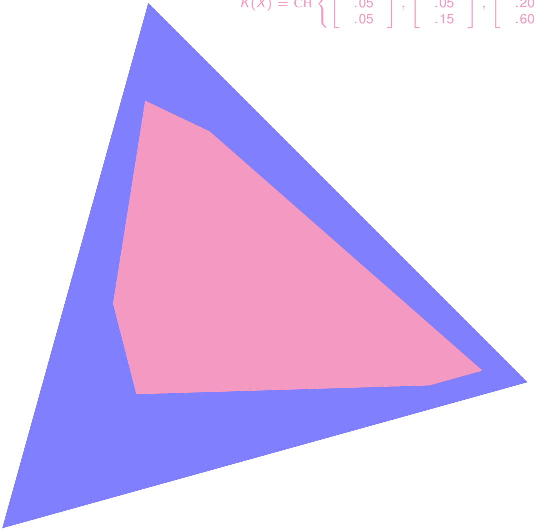
Probability intervals are not fully general

$$K(X) = \mathrm{CH}\left\{ \begin{bmatrix} .90 \\ .05 \\ .05 \end{bmatrix}, \begin{bmatrix} .80 \\ .05 \\ .15 \end{bmatrix}, \begin{bmatrix} .20 \\ .20 \\ .60 \end{bmatrix}, \begin{bmatrix} .10 \\ .40 \\ .50 \end{bmatrix}, \begin{bmatrix} .05 \\ .80 \\ .15 \end{bmatrix}, \begin{bmatrix} .20 \\ .70 \\ .10 \end{bmatrix} \right\}$$



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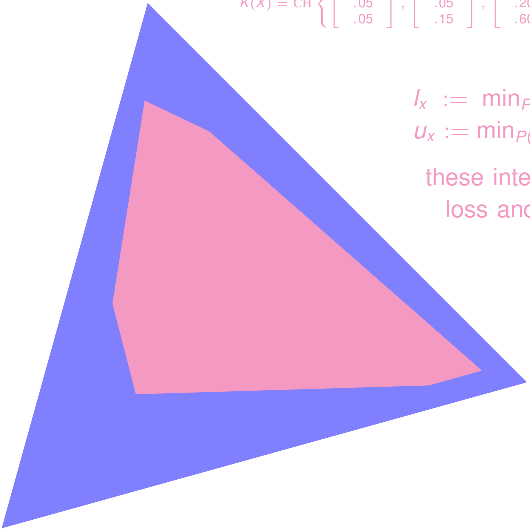

$$l_x := \min_{P(X) \in K(X)} p(x)$$
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these intervals avoid sure loss and are coherent

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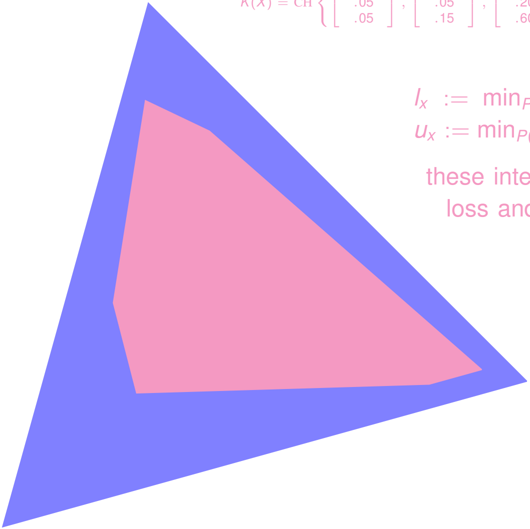

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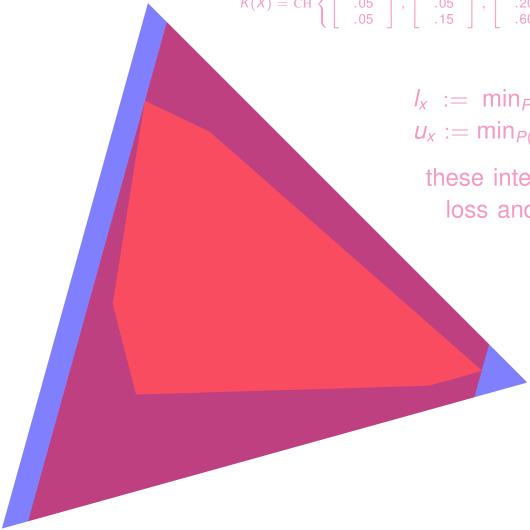

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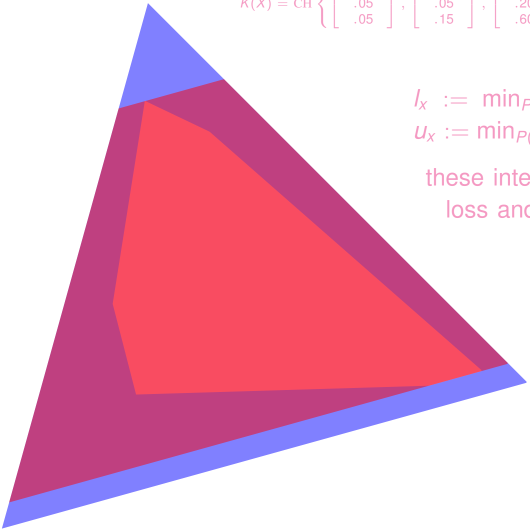

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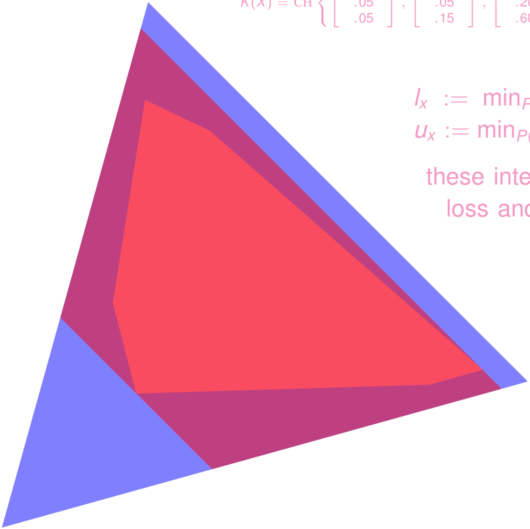

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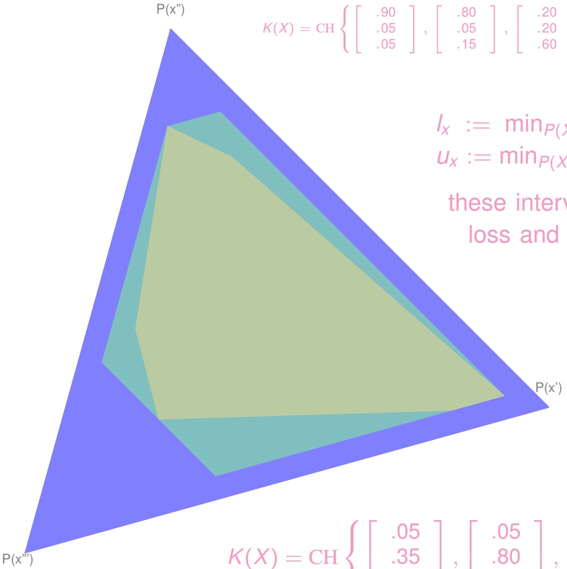

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Learning credal sets from (few) data

- Learning from data about X
- Max lik estimate $P(x) = \frac{n(x)}{N}$
- Bayesian (ESS $s = 2$) $\frac{n(x)+st(x)}{N}$
- Imprecise: set of priors (vacuous t)

$$\frac{n(x)}{N+s} \leq P(x) \leq \frac{n(x)+s}{N+s}$$

imprecise Dirichlet model
(Walley & Bernard)

- They a.s.l. and are coherent
- Non-negligible size of intervals only for small N (Bayesian for $N \to \infty$)

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$$\begin{array}{r} n(\mathit{win}) \\ n(\mathit{draw}) \\ n(\mathit{loss}) \end{array} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

1957: Spain vs. Italy 5 — 1
1973: Italy vs. Spain 3 — 2
1980: Spain vs. Italy 1 — 0
1983: Spain vs. Italy 1 — 0
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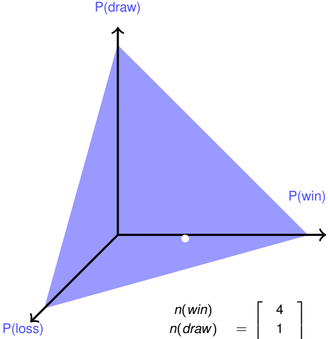

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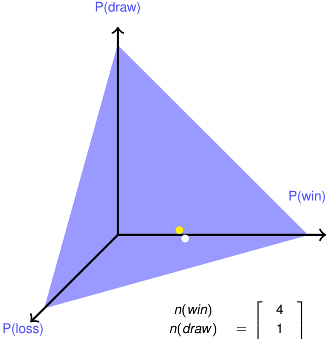

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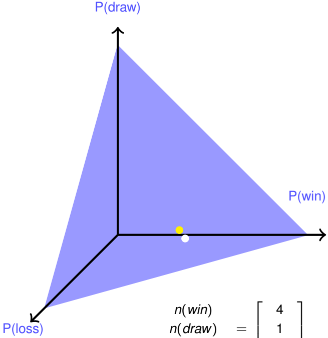
Learning credal sets from (few) data

- Learning from data about X
- Max lik estimate $P(x) = \frac{n(x)}{N}$
- Bayesian (ESS $s = 2$) $\frac{n(x)+st(x)}{N}$
- Imprecise: set of priors (vacuous t)

$$\frac{n(x)}{N+s} \leq P(x) \leq \frac{n(x)+s}{N+s}$$

imprecise Dirichlet model
(Walley & Bernard)

- They a.s.l. and are coherent
- Non-negligible size of intervals only for small N (Bayesian for $N \to \infty$)



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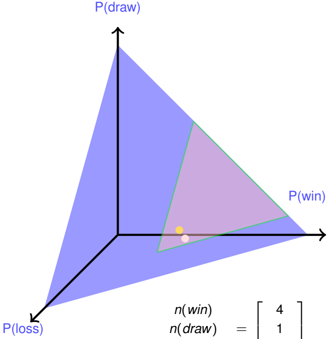

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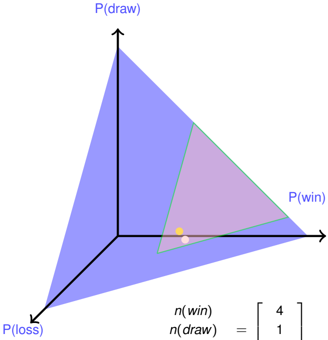

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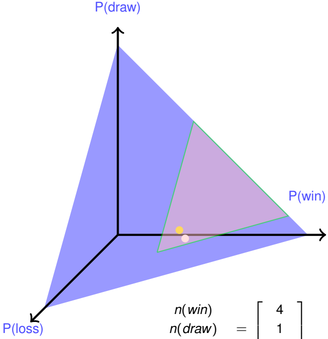

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Learning credal sets from (missing) data

- Coping with missing data?
- Missing at random (MAR)
  $P(O = *|X = x)$ indep of X
  Ignore missing data
- Not always the case!
- Conservative updating
  (Gert & Zaffalon) ignorance about the process $P(O|X)$ as a vacuous model
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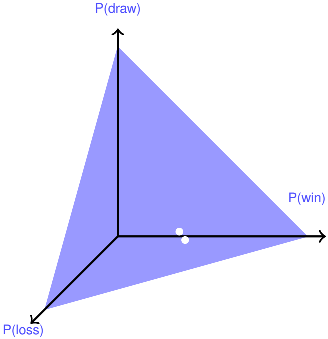

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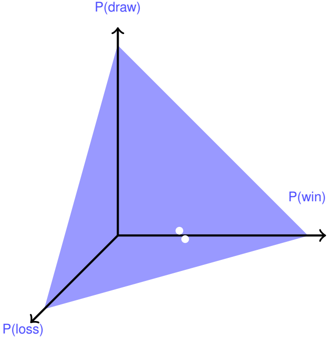

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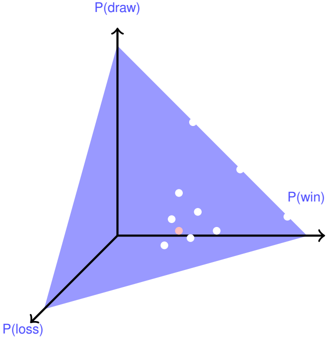

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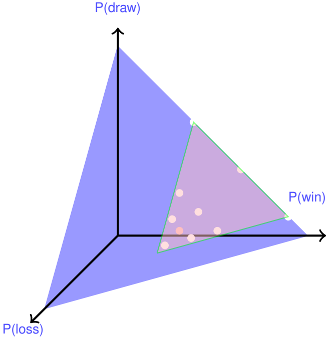

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