# Independence (and Some Other Structural Assessments) in Imprecise Probability

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July 16, 2012

# Overview

1. A review of some basic definitions.
2. Structural assessments: vacuity, uniformity, exchangeability.
3. A review of stochastic (conditional) independence.
4. Confirmational/complete/strong/epistemic/Kuznetsov/others independence.
5. Comparison.
6. A look into the messy world of zero probabilities.

# Easy warm-up

- Possibility space $\Omega$ with states $\omega$.
- Events are subsets of $\Omega$.
- Random variables ($X : \Omega \to \Re$).
- Expectations $E[X]$, probabilities $P(A)$.
- Conditional expectations $E[X|B]$, probabilities $P(A|B)$.

# Credal sets

- A *credal set* is a set of probability measures (distributions).
- A credal set is usually defined by a set of *assessments*.

Example:

1. $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
2. $P(\omega_i) = p_i$.
3. $p_1 \geq p_3$, $2p_1 \geq p_2$, $p_1 \leq 2/3$ and $p_3 \in [1/6, 1/3]$.
4. Take points $P = (p_1, p_2, p_3)$.

# Some geometry

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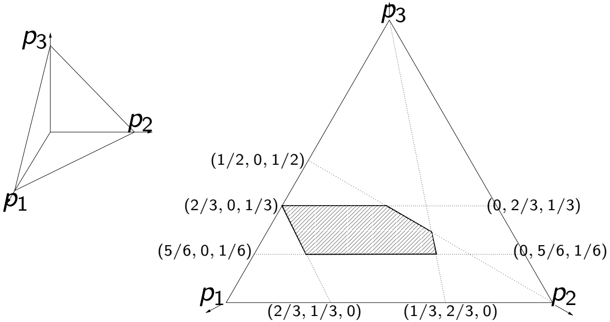

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- For closed convex credal sets, lower and upper expectations are attained at vertices.
- A closed convex credal set is completely characterized by the associated lower expectation.
  - That is, there is only one lower expectation for a given closed convex credal set.

# Generating credal sets I

From partial preferences:

- $X \succ Y$ means "$X$ is preferred to $Y$."
- Axiomatize $\succ$ as partial order.
- Then:

$$X \succ Y \quad \text{iff} \quad E_P[X] > E_P[Y] \text{ for all } P \in K.$$

- Credal sets with identical vertices produce the same $\succ$.
- Focus has been on unique *maximal* credal set that represents $\succ$.
  - Smaller credal sets have no "behavioral" significance.

# Generating credal sets II

From one-sided betting:

- Variables are *gambles*.
- Buy/sell gambles using $\underline{E}[X]$ and $\overline{E}[X]$.
- Some constraints, such as: $\sum_{i=1}^{n} X_i - \underline{E}[X_i] \geq 0$.
- Credal set is produced by the set of *dominating* expectations:

$$\{E : E[X] \geq \underline{E}[X]\}.$$

- Several credal sets produce the same lower expectations.
  - But only maximal closed one is given "behavioral" significance.

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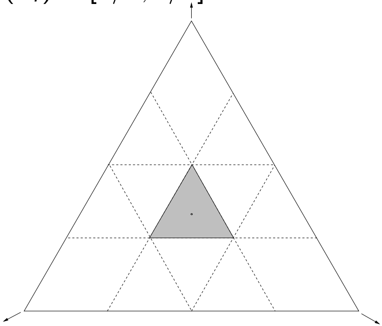
# Structural assessments

- A structural assessment constrains a large (possibly infinite) number of expectations.

- A simple example: vacuity.

# Uniformity

- Every $\omega$ is subject to identical assessments.
- Extreme case: vacuity.
- Extreme case: uniform distribution.
- Intermediate case: $P(\omega_i) \in [1/4, 1/2]$.



# Exchangeability

- A basic structural assessment.
- To simplify, take categorical variables $\mathbf{X} = [X_1, \ldots, X_m]$.

- Denote by $\pi_m$ a permutation of integers $\{1, \ldots, m\}$, and by $\pi_m(i)$ the $i$th number in the permutation.
- Denote

$$\{\mathbf{X} = \mathbf{x}\} \doteq \cap_{i=1}^{m} \{X_i = x_i\},$$

and

$$\{\pi_m \mathbf{X} = \mathbf{x}\} \doteq \cap_{i=1}^{m} \{X_{\pi_m(i)} = x_i\}.$$

# Definition of exchangeability

- Variables $X_1, \ldots, X_m$ are *exchangeable* when

$$\underline{E}[\{\mathbf{X} = \mathbf{x}\} - \{\pi_m \mathbf{X} = \mathbf{x}\}] = 0$$

  for any permutation $\pi_m$.

- That is, the order of variables does not matter: trading $\{\mathbf{X} = \mathbf{x}\}$ for $\{\pi_m \mathbf{X} = \mathbf{x}\}$ does not seem advantageous.

- Walley's theorem: this implies *elementwise* exchangeability.

# Walley's exchangeability theorem

- In words: Exchangeability implies *elementwise* exchangeability.
- Proof:

$$
\begin{aligned}
0 &= \underline{E}[\{\mathbf{X} = \mathbf{x}\} - \{\pi_m \mathbf{X} = \mathbf{x}\}] \\
&\leq \overline{E}[\{\mathbf{X} = \mathbf{x}\} - \{\pi_m \mathbf{X} = \mathbf{x}\}] \\
&= -\underline{E}[\{\pi_m \mathbf{X} = \mathbf{x}\} - \{\mathbf{X} = \mathbf{x}\}] = 0.
\end{aligned}
$$

Hence every distribution in the credal set $K(X_1, \ldots, X_m)$ satisfies

$$P(\mathbf{X} = \mathbf{x}) = P(\pi_m \mathbf{X} = \mathbf{x}) \quad \text{for any permutation } \pi_m.$$

# Exercise

- Suppose we have 4 binary variables that are exchangeable.
- What are the conditions on the probabilities $P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4)$?

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Here they are:

- One success: $P(0001) = P(0010) = P(0100) = P(1000)$.
- Two successes: $P(1001) = P(1010) = P(1100) = P(0101) = P(0110) = P(0011)$.
- Three successes: $P(1110) = P(1101) = P(1011) = P(0111)$.

# Facts about exchangeability

- For $X_1, \dots, X_m$, what matters is

$$P\left(\sum_{i=1}^{m} X_i = r\right).$$

- Any subset of exchangeable variables is exchangeable.
- Consider $m$ exchangeable variables, and take initial $n$ variables; then $P(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0)$ is equal to

$$\sum_{r=k}^{m-n+k} \frac{\binom{m-n}{r-k}}{\binom{m}{r}} P\left(\sum_{i=1}^{n} X_i = r\right).$$

# de Finetti's theorem (binary variables)

- Denote $P(X_1 = 1)$ by $\theta$ and take $m \to \infty$:
  Then $P(X_1 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_n = 0)$ is equal to
  $$\int_0^1 \theta^k (1-\theta)^{n-k} dF(\theta).$$
- That is, behaves like $\prod_{i=1}^n P(X_i = 1)$ with a (yet unspecified) distribution over $\theta$.

# Exercise

Draw the credal set $K(X, Y)$ for binary variables $X$ and $Y$ given the structural assessments:

1. $X$ and $Y$ are exchangeable.
2. $X$ and $Y$ are the first two variables in a sequence of three exchangeable variables.
3. $X$ and $Y$ are the first two variables in a sequence of five exchangeable variables.
4. $X$ and $Y$ are the first two variables in a sequence of infinitely many exchangeable variables.

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# Independence

- Events:
  - $P(A|B) = P(A)$.
  - $P(A \cap B) = P(A)\, P(B)$.
- Variables:
  - $E[f(X)|A(Y)] = E[f(X)]$.
  - $E[f(X)g(Y)] = E[f(X)]\, E[g(Y)]$.

# Independence for events

- $A$ and $B$ are independent

$$P(A|B) = P(A) \qquad \text{whenever } P(B) > 0;$$

or, equivalently (definition is symmetric!),

$$P(A \cap B) = P(A)\, P(B)\,.$$

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- For all subsets of events $\{A_i\}_{i=1}^{n}$,

$$P(\cap_i \{X_i \in A_i\}) = \prod_i P(\{X_i \in A_i\})\,.$$

# Independence for variables

- $X$ is stochastically irrelevant to $Y$ when:

$$E[f(Y)|\{X \in A\}] = E[f(Y)]$$

for any bounded function $f$, whenever $P(\{X \in A\}) > 0$.

# Independence for variables

- $X$ is stochastically irrelevant to $Y$ when:

  $$E[f(Y)|\{X \in A\}] = E[f(Y)]$$

  for any bounded function $f$, whenever $P(\{X \in A\}) > 0$.
- Definition is symmetric! We might as well use the symmetric condition:

  $$E[f(X)g(Y)] = E[f(X)]\,E[g(Y)]$$

  for every bounded $f$ and $g$.
- So, take these conditions to mean the symmetric concept of *stochastic independence* of $X$ and $Y$.

# Other definitions...

1. $X$ is stochastically irrelevant to $Y$ when

$$P(\{Y \in B\}|\{X \in A\}) = P(\{Y \in B\})$$

   whenever $P(\{X \in A\}) > 0$.
2. $X$ is stochastically irrelevant to $Y$ when

$$P(\{Y \in B\} \cap \{X \in A\}) = P(\{Y \in B\})\, P(\{X \in A\}) \, .$$

# For many variables...

- Variables $\{X_i\}_{i=1}^n$ are *independent* if

$$E[f_i(X_i)| \cap_{j\neq i} \{X_j \in A_j\}] = E[f_i(X_i)]\,,$$

  for

  - all functions $f_i(X_i)$,
  - all events $\cap_{j\neq i}\{X_j \in A_j\}$ with positive probability.

- For all functions $f_i(X_i)$,

$$E\left[\prod_{i=1}^n f_i(X_i)\right] = \prod_{i=1}^n E[f_i(X_i)]\,.$$

# Conditional stochastic independence

- All these definitions can be extended to *conditional independence by conditioning on some $Z$, for every value of $Z$.*

- *For instance, $X$ and $Y$ are conditionally stochastically independent* given $Z$ iff

$$\begin{aligned} P(\{Y \in B\} \cap \{X \in A\} | Z = z) &= P(\{Y \in B\} | Z = z) \\ &\quad \times P(\{X \in A\} | Z = z) \end{aligned}$$

whenever $P(Z = z) > 0$.

# What do we do with independence?

- We consider repetitions of experiments.
- We prove convergence under quite broad conditions.
- We construct large models by combining pieces (see exciting talk tomorrow on credal networks!!).
- We discard parts of a model when focusing on particular inferences.

# Laws of large numbers...

- Weak law of large numbers:

$$\forall \epsilon > 0, \quad \lim_{n \to \infty} P\big(|\bar{X} - \mu| \geq \epsilon\big) = 0$$

- Strong law of large numbers:

$$P\left(\lim_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{n} = \mu\right) = 1.$$

- (Finitely-additive) strong law of large numbers: for all $\epsilon > 0$, there is integer $N$ such that for every positive integer $k$,

$$P\left(\exists n \in [N, N+k] : \left|\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right| > \epsilon\right) < \epsilon.$$

# The graphoid properties

Proposed as a way to encode the intuitive meaning of "independence".
Actually, refer to "conditional independence".

Symmetry: $(X \perp\!\!\!\perp Y \mid Z) \Rightarrow (Y \perp\!\!\!\perp X \mid Z)$

Decomposition: $(X \perp\!\!\!\perp (W, Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid Z)$

Weak union: $(X \perp\!\!\!\perp (W, Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp W \mid (Y, Z))$

Contraction: $(X \perp\!\!\!\perp Y \mid Z)$ & $(X \perp\!\!\!\perp W \mid (Y, Z)) \Rightarrow (X \perp\!\!\!\perp (W, Y) \mid Z)$

Redundancy: $(X \perp\!\!\!\perp Y \mid X)$

Often added (true when probabilities are positive):

Intersection $(X \perp\!\!\!\perp W \mid (Y, Z))$ & $(X \perp\!\!\!\perp Y \mid (W, Z)) \Rightarrow (X \perp\!\!\!\perp (W, Y) \mid Z)$

Satisfied by many structures (graphs, lattices, etc).

# Exercise

Prove contraction for stochastic independence.

Contraction: $(X \perp\!\!\!\perp Y \,|\, Z)$ & $(X \perp\!\!\!\perp W \,|\, (Y, Z)) \Rightarrow$
$(X \perp\!\!\!\perp (W, Y) \,|\, Z)$

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# Complete independence

- $X$ and $Y$ are *completely independent* if for all $P \in K(X, Y)$,

$$P(X \in A | Y \in B) = P(X \in A) \qquad \text{whenever } P(Y \in B) > 0.$$

- That is, elementwise stochastic independence.

- This concept violates convexity (presumably has no "behavioral" justification).

# Failure of convexity

Example of Jeffrey's:

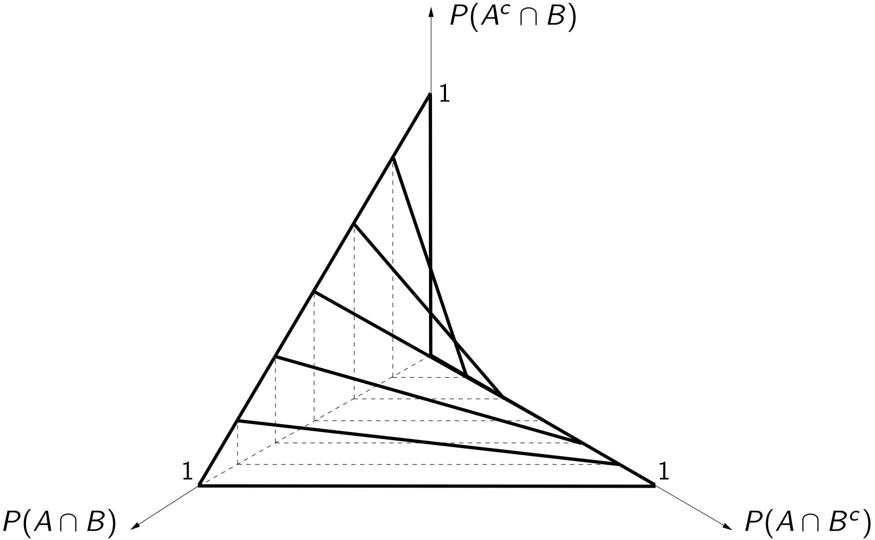
- Binary variables $X$ and $Y$, completely independent.
- $K(X, Y)$: convex hull of $P_1$ and $P_2$,

$$P_1(X = 0) = P_1(Y = 0) = 1/3, P_2(X = 0) = P_2(Y = 0) = 2/3.$$

- Take $P_{1/2} = P_1/2 + P_2/2$ (by convexity, $P_{1/2} \in K(X, Y)$).
- However,

$$\begin{aligned} P_{1/2}(X = 0, Y = 0) &= P_1(X = 0)P_1(Y = 0)/2 + \\ &\quad P_2(X = 0)P_1(Y = 0)/2 \\ &= 5/18 \neq 1/4 \\ &= P_{1/2}(X = 0)P_{1/2}(Y = 0). \end{aligned}$$

# Independence surface for two events



# Confirmational independence

- I. Levi, the pioneer on convex credal sets, detected this problem with complete independence.

- His proposal: $Y$ is *confirmationally irrelevant* to $X$ if

$$K(X|Y \in B) = K(X) \qquad \text{for nonempty } \{Y \in B\},$$

- His position: use complete independence if needed, but take convex hull (does not affect partial preferences...).

# Strong independence

- $X$ and $Y$ are *strongly independent* when $K(X, Y)$ is the convex hull of a set of distributions satisfying complete independence.

# Type-1, type-2 products and others

- Walley and Fine (1982) called this expression an *independent product* when restricted to indicators:

$$\underline{E}[A(X,Y)] = \min\left(E_P[A(X,Y)] : P = P_X P_Y\right).$$

- This is Weichselberger's definition of *mutual independence*.
- In his book, Walley (1991) called the general expression a *type-1 product*.
- ...and *type-2 products* refer to the case of identical marginals.

# Epistemic irrelevance

- Walley also proposes a different concept: $Y$ is epistemically irrelevant to $X$ if for any bounded function $f(X)$,

$$\underline{E}[f(X)|Y \in B] = \underline{E}[f(X)] \quad \text{for nonempty } \{Y \in B\}.$$

- *If* credal sets are closed and convex, then epistemic irrelevance is identical to Levi's confirmational irrelevance.

# Exercise (Couso et al 1999, Example 3)

- Consider three urns (with red and white balls):

| Urn | Red | White | Unknown |
|---|---|---|---|
| A | 5 | 2 | 3 |
| B | 3 | 3 | 4 |
| C | 3 | 3 | 4 |

- Take ball $X$ from A, then
  - if red, take ball $Y$ from B,
  - otherwise take ball $Y$ from C.
- How to construct set of joint distributions for $(X, Y)$?
- What are $P(Y = R|X = R)$, $P(Y = R|X = W)$, $P(Y = R)$?
- But notice: $P(X = R|Y = R) \in [3/10, 28/31]$ (symmetry fails).

# Epistemic independence

- Walley's clever idea: "symmetrize" irrelevance (this is actually a strategy by Keynes).

- $X$ and $Y$ are *epistemically independent* if $Y$ is epistemically irrelevant to $X$ and $X$ is epistemically irrelevant to $Y$.

- Quite an intuitive concept that "generates convexity" automatically.

# Strong $\neq$ Epistemic

- Two binary variables $X$ and $Y$.
- $P(X = 0) \in [2/5, 1/2]$ and $P(Y = 0) \in [2/5, 1/2]$.
- Epistemic independence of $X$ and $Y$: $K(X, Y)$ is convex hull of

$$[1/4, 1/4, 1/4, 1/4], [4/25, 6/25, 6/25, 9/25],$$

$$[1/5, 1/5, 3/10, 3/10], [1/5, 3/10, 1/5, 3/10],$$

$$[2/9, 2/9, 2/9, 1/3], [2/11, 3/11, 3/11, 3/11],$$

# Kuznetsov: some interval arithmetic

- Kuznetsov (1991) proposed yet another concept.
- Denote by $EI[X]$ the interval $\left[\underline{E}[X], \overline{E}[X]\right]$.
- $X$ and $Y$ are *Kuznetsov independent* if, for any bounded functions $f(X)$ and $g(Y)$,

$$EI[f(X)g(Y)] = EI[f(X)] \times EI[g(Y)].$$

# Exercise

Prove:

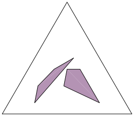
- Kuznetsov independence implies epistemic independence (assume all probabilities are nonzero!).
- Epistemic independence does not imply Kuznetsov independence.

# Strong $\neq$ Kuznetsov

- It would be nice if Kuznetsov and strong independence were equivalent.
- But they are not!
- (Actually, they are equivalent if one of the variables is binary.)

# Example

- Ternary variables $X$ and $Y$, credal sets $K(X)$ and $K(Y)$:



- Largest set that satisfies strong independence (strong extension) has 16 vertices and 24 facets; for instance, a facet with normal

$$[-434, 301, 21, 2836, -1154, -1734, -1164, 96, 1116].$$

- This facet cannot be written as $f(X)g(Y) + \alpha$.
- Intuitively, a Kuznetsov "extension" wraps the strong extension using only functions $f(X)g(Y)$.

# Some history

- Several variants between 1990/2000... inspired by intense activity in Dempster-Shafer and possibility theory.
- For each possible definition of conditioning or product-measure, a concept of independence...
  - Quick example: Dempster conditioning defines

$$\overline{P}(X|_D Y) = \overline{P}(X, Y)\,/\overline{P}(Y)$$

then we can impose

$$\overline{P}(X|_D Y) = \overline{P}(X, Y)\,/\overline{P}(Y) = \overline{P}(X)\,.$$

  - Related (mathematically at least) to Shafer's concept of *cognitive independence*

# de Campos and Moral, 1995

- Attempt to organize the field.
- Their *type-2* independence is strong independence.
- Their *type-3* independence obtains when $K(X, Y)$ is the convex hull of *all* product distributions $P_X P_Y$, where $P_X \in K(X)$ and $P_Y \in K(Y)$.
  - That is, type-3 independence is simply strong extension.
- Their *type-5* independence is a variant on confirmational irrelevance.

# Type-5 independence

- $Y$ is *type-5* irrelevant to $X$ if

$$R(X|Y \in B) = K(X) \qquad \text{whenever } \overline{P}(Y \in B) > 0,$$

where $R(X|Y \in B)$ denotes the set

$$\{P(\cdot|Y \in B) : P \in K(X, Y)\,; P(Y \in B) > 0\}.$$

- Then take *type-5 independence* to be the "symmetrized" concept.
- The set $R$ is often used to define conditioning (related to what Walley calls *regular extension*).

# Exercise

Due to de Campos and Moral (1995).

- $X$ and $Y$ are binary.
- $K(X, Y)$ is the convex hull of two distributions $P_1$ and $P_2$ such that $P_1(X = 0, Y = 0) = P_2(X = 1, Y = 1) = 1$.

Show:

- $X$ and $Y$ are strongly independent.
- Neither $Y$ is type-5 irrelevant to $X$, nor $X$ is type-5 irrelevant to $Y$.

# Couso et al, 1999

- In 1999 Couso et al presented an influential review.
  - Their *independence in the selection* is strong independence.
  - Their *strong independence* is strong extension.
  - Their *repetition independence* refers to Walley's *type-2 product*.

- They also discuss *non-interactivity* and *random set independence* (called *belief function product* by Walley and Fine, 1982).

# The zoo, so far

- Complete independence.
- Confirmational and epistemic irrelevance/independence.
- Strong independence.
- Kuznetsov independence.
- Type-5 independence.

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Comments:

- Epistemic independence is most intuitive (under convexity).
- Complete independence is closer to stochastic independence (without convexity).
- How to justify strong independence?

# Conditional independence

- Any concept of independence can be modified to express *conditional independence*.
- For example, *conditional* epistemic irrelevance of $Y$ to $X$ given $Z$:

$$\underline{E}[f(X)|Y \in B, Z = z] = \underline{E}[f(X)|Z = z]$$

  for all bounded functions $f(X)$ and all nonempty $\{Z = z\}$.
- Likewise for conditional Kuznetsov/complete/strong independence of $X$ and $Y$ given $Z$.
- Complications with probability zero, but let's postpone that...

# Overview

1. Some basic (mostly known) definitions: credal sets, lower expectations and probabilities, decision making, and the like.
2. Structural assessments: vacuity, uniformity, exchangeability.
3. A brief review of stochastic (conditional) independence.
4. Confirmational/complete/strong/epistemic/Kuznetsov/others independence.
5. **Comparison.**
6. A look into the messy world of zero probabilities.

# Comparing concepts

There are perhaps too many concepts around.

- Idea: verify which concepts satisfy laws of large numbers.
  - Not really discriminating: all satisfy forms of laws of large numbers (results by de Cooman and Miranda).
- Other idea: check graphoid properties.

# Reminder: graphoid properties

Symmetry: $(X \perp\!\!\!\perp Y \,|\, Z) \Rightarrow (Y \perp\!\!\!\perp X \,|\, Z)$

Decomposition: $(X \perp\!\!\!\perp (W, Y) \,|\, Z) \Rightarrow (X \perp\!\!\!\perp Y \,|\, Z)$

Weak union: $(X \perp\!\!\!\perp (W, Y) \,|\, Z) \Rightarrow (X \perp\!\!\!\perp W \,|\, (Y, Z))$

Contraction: $(X \perp\!\!\!\perp Y \,|\, Z)$ & $(X \perp\!\!\!\perp W \,|\, (Y, Z)) \Rightarrow$ $(X \perp\!\!\!\perp (W, Y) \,|\, Z)$

# Exercise

Show that complete and strong independence satisfy all graphoid properties.

# Failure of contraction

- Epistemic independence satifies symmetry, redundancy, decomposition, weak union, but fails contraction even when all probabilities are positive.
  - Thus type-5 independence also fails contraction.
- Kuznetsov independence fails contraction even when all probabilities are positive.

Note: there are different results when probabilities can be equal to zero!

# Failure of contraction: epistemic indep.

- Binary variables $W$, $X$ and $Y$.
- $K(W, X, Y)$ is convex hull of three distributions:

| $W$ | $X$ | $Y$ | $p_1(X, Y, W)$ | $p_2(X, Y, W)$ | $p_3(X, Y, W)$ |
|---|---|---|---|---|---|
| $W_0$ | $X_0$ | $Y_0$ | 0.008 | 0.018 | 0.0093 |
| $W_1$ | $X_0$ | $Y_0$ | 0.072 | 0.072 | 0.0757 |
| $W_0$ | $X_1$ | $Y_0$ | 0.032 | 0.042 | 0.037 |
| $W_1$ | $X_1$ | $Y_0$ | 0.288 | 0.168 | 0.228 |
| $W_0$ | $X_0$ | $Y_1$ | 0.096 | 0.084 | 0.09 |
| $W_1$ | $X_0$ | $Y_1$ | 0.024 | 0.126 | 0.075 |
| $W_0$ | $X_1$ | $Y_1$ | 0.384 | 0.196 | 0.290 |
| $W_1$ | $X_1$ | $Y_1$ | 0.096 | 0.294 | 0.195 |

- $X$ and $Y$ are epistemically independent; $X$ and $W$ are conditionally epistemically independent given $Y$.
- But $X$ and $(W, Y)$ are not not epistemically independent.

# Failure of contraction: Kuznetsov indep.

- Binary variables $W$, $X$, and $Y$
- $K(W, X, Y)$ with four vertices (each is the product of $p(W|Y)\, p(Y)\, p(X)$):

| Vertex | $p_i(w_0\|y_0)$ | $p_i(w_0\|y_1)$ | $p_i(x_0)$ | $p_i(y_0)$ |
|---|---|---|---|---|
| $p_1$ | 0.7 | 0.4 | 0.2 | 0.2 |
| $p_2$ | 0.7 | 0.4 | 0.3 | 0.3 |
| $p_3$ | 0.8 | 0.5 | 0.2 | 0.3 |
| $p_4$ | 0.8 | 0.5 | 0.3 | 0.2 |

- $X$ and $Y$ are Kuznetsov independent; $X$ and $W$ are conditionally Kuznetsov independent given $Y$.
- But $X$ and $(W, Y)$ are not Kuznetsov independent.

# Exercise

- Show that epistemic irrelevance satisfies: if $Y$ is epistemically irrelevant to $X$ and $W$ is epistemically irrelevant to $X$ given $Y$ then $(W, Y)$ are epistemically irrelevant to $X$.

# Complexity

- Little is known about the computational complexity of various concepts.
- Complete/strong independence have been addressed in the context of credal networks.
- Some algorithms are known for epistemic independence.
- It seems that complete/strong independence are “more tractable” in an informal way.

# The zoo, so far...

- Complete independence.
- Confirmational and epistemic irrelevance/independence.
- Strong independence.
- Kuznetsov independence (not very promising).
- Type-5 independence (only relevant with zero probabilities).

Comments:

- Epistemic independence is more intuitive (under convexity).
- Complete independence is closer to stochastic independence (without convexity).
- How to justify strong independence?

# Justifying strong independence

- Sensitivity analysis interpretation: several experts agree on stochastic independence.
  - This is an argument for complete independence.

- Is there a justification that uses partial preferences, lower expectations, credal sets, etc?

- A possible idea: changes in assessments (Cozman (2000), Moral and Cano (2002)).

# Producing strong independence

- Moral and Cano (2002):

  Variables $X$ and $Y$ are [fully] strongly independent iff they are epistemically independent after $K(X, Y)$ is combined with an arbitrary collection of compatible assessments on $X$ and on $Y$.
- Compatibility requires some maneuvers "similar to" Jeffrey's rule: we change the marginal, then see what happens to the other marginal.

# Another justification: exchangeability

- Consider a vector of $m$ exchangeable binary variables $\mathbf{X} = [X_1, \ldots, X_m]$.
- If we look at the first $n$ variables and let $m \to \infty$, then $P(X_1 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_n = 0)$ is

$$\int_0^1 \theta^k (1 - \theta)^{n-k} dF(\theta).$$

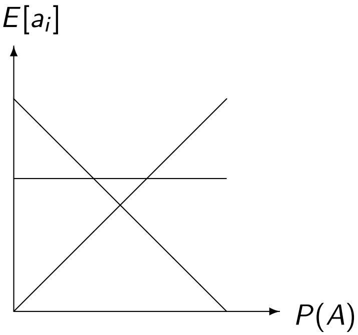
- Remember: $\theta$ is the probability of $\{X_1 = 1\}$.
- We have a convex credal set $K(\theta)$.
- Strong independence obtains if each vertex of $K(\theta)$ assigns probability 1 to a particular value of $\theta$.
- We have in fact obtained a type-2 product.
- Similar argument works for general variables.

# Back to complete independence

- Complete independence is very attractive.
- But it violates convexity.
- So, it does not have a "behavioral" interpretation...
- Is it true?
- NO!

# Seidenfeld cuts

Three acts: $a_1 = 0.6$; $a_2 = 0/1$ if $A/A^c$; $a_3 = 1/0$ if $A/A^c$.




We can "cut" pieces of the probability interval!

# That is,

*There is a difference between a set of distributions and its convex hull when one chooses among several acts.*

# Axiomatizing partial preferences

- Can we axiomatize preferences amongst sets of acts, so as to obtain general credal sets?

- Yes. It has been done by Seidenfeld et al (2007) [it seems first idea by Kyburg and Pittarelli (1992)].

# Producing complete independence

- Are events $A$ and $B$ are completely independent?
- Construct five acts $a_0, \ldots, a_4$:

| | $AB$ | $A^cB$ | $AB^c$ | $A^cB^c$ |
|---|---|---|---|---|
| $a_0$ | $0$ | $0$ | $0$ | $0$ |
| $a_1$ | $1-\alpha$ | $-\alpha$ | $0$ | $0$ |
| $a_2$ | $-(1-\alpha)$ | $\alpha$ | $0$ | $0$ |
| $a_3$ | $0$ | $0$ | $1-\beta$ | $-\beta$ |
| $a_4$ | $0$ | $0$ | $-(1-\beta)$ | $\beta$ |

- Test: if we observe that for every $\alpha, \beta \in (0,1)$ such that $\alpha \neq \beta$ we have some act rejected, we can conclude that $A$ and $B$ are completely independent.

# Overview

1. Some basic (mostly known) definitions: credal sets, lower expectations and probabilities, decision making, and the like.
2. Structural assessments: vacuity, uniformity, exchangeability.
3. A brief review of stochastic (conditional) independence.
4. Confirmational/complete/strong/epistemic/Kuznetsov/others independence.
5. Comparison.
6. **A look into the messy world of zero probabilities.**

# Potentially null events

- Events may have zero *lower* probability but nonzero *upper* probability (cannot ignore those).
- Example of difficulties one may face:
  - Suppose we refuse to define $K(X|Y=y)$ when $\underline{P}(Y=y)=0$.
  - Now consider the following concept: $Y$ is "irrelevant" to $X$ if

$$K(X|Y \in B) = K(X) \quad \text{whenever } \underline{P}(Y \in B) > 0.$$

  - But this is quite weak:
    - We may have $\underline{P}(Y \in B) = 0$ for every $B \neq \Omega$;
    - then $Y$ is irrelevant to any other variable!

# Full conditional measures

- The most elegant solution is to consider *full probability measures*.
- A full probability measure is a function $P(\cdot|\cdot)$ on $\mathcal{E} \times \mathcal{E} \backslash \emptyset$ where $\mathcal{E}$ is an algebra of events, such that
  - $P(\Omega|C) = 1$;
  - $P(A|C) \geq 0$ for all $A$;
  - $P(A \cup B|C) = P(A|C) + P(B|C)$ when $A \cap B = \emptyset$;
  - $P(A \cap B|C) = P(A|B \cap C)\, P(B|C)$ when $B \cap C \neq \emptyset$.
- Full probability measures allow $P(A|C)$ to be defined even if $P(C) = 0$!

# The Krauss-Dubins representation

- We can partition a $\Omega$ into events $L_0, \ldots, L_K$, where $K \leq N$,
- such that the full conditional measure is represented as a sequence of completely positive probability measures $P_0, \ldots, P_K$, where the support of $P_i$ is restricted to $L_i$.

Example:

| | $A$ | $A^c$ |
|---|---|---|
| $B$ | 0 | $\alpha$ |
| $B^c$ | 0 | $1-\alpha$ |

| | $A$ | $A^c$ |
|---|---|---|
| $B$ | $\beta$ | |
| $B^c$ | $1-\beta$ | |

Left: $P$. Right: $P(\cdot|A)$.
Here: $P(A) = 0$, but $P(B|A) = \beta$.

# Using full conditional measures

- Unsurprisingly, Levi and Walley both adopt full conditional measures.
- A challenge is that full conditional measures seem to call for finite additivity.
    - Again, this is the path taken by Levi and Walley.

# A problem with stochastic independence

- The usual product definition is now too weak!
- Consider: we may have

$$P(X, Y = y|Z = z) = P(X|Z = z)\, P(Y = y|Z = z)$$

  and yet

$$P(X|Y = y, Z = z) \neq P(X|Z = z)\,.$$

- (Failure may happen when $P(Y = y, Z = z) = 0$.)

# Failure of symmetry

- Take epistemic irrelevance:

$$P(X|Y = y, Z = z) = P(X|Z = z)\,.$$

- But: this is not symmetric!!

Example:

| | $A$ | $A^c$ |
|---|---|---|
| $B$ | $\lfloor\beta\rfloor_1$ | $\alpha$ |
| $B^c$ | $\lfloor 1-\beta\rfloor_1$ | $1-\alpha$ |

Note: $P(A|B) = P(A)$, but $P(B|A) \neq P(B)$!

# As before: symmetrize!

- Definition of *epistemic* independence:
  Require
  $$P(X|Y = y, Z = z) = P(X|Z = z)$$
  and
  $$P(Y|X = x, Z = z) = P(Y|Z = z) \, .$$
- This is symmetric for sure.
- How does it fare with respect to graphoid properties?

# Reminder: graphoid properties

Symmetry: $(X \perp\!\!\!\perp Y \mid Z) \Rightarrow (Y \perp\!\!\!\perp X \mid Z)$

Decomposition: $(X \perp\!\!\!\perp (W, Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid Z)$

Weak union: $(X \perp\!\!\!\perp (W, Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp W \mid (Y, Z))$

Contraction: $(X \perp\!\!\!\perp Y \mid Z)$ & $(X \perp\!\!\!\perp W \mid (Y, Z)) \Rightarrow$
$(X \perp\!\!\!\perp (W, Y) \mid Z)$

# Problem with epistemic independence

- It fails weak union!

| | $w_0 y_0$ | $w_1 y_0$ | $w_0 y_1$ | $w_1 y_1$ |
|---|---|---|---|---|
| $x_0$ | $\alpha$ | $\lfloor \beta \rfloor_2$ | $1-\alpha$ | $\lfloor 1-\beta \rfloor_2$ |
| $x_1$ | $\lfloor \alpha \rfloor_1$ | $\lfloor \gamma \rfloor_3$ | $\lfloor 1-\alpha \rfloor_1$ | $\lfloor 1-\gamma \rfloor_3$ |

Remember:

Weak union: $(X \perp\!\!\!\perp (W,Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid (W,Z))$

# Hammond's independence

- Here is a proposal for independence:

$$P(B(Y)|A(X) \cap D(Y)) = P(B(Y)|D(Y)) \text{ and}$$

$$P(A(X)|B(Y) \cap C(X)) = P(A(X)|C(X)) .$$

- This is symmetric.
- It satisfies weak union! But if fails contraction...

Remember:

Contraction: $(X \perp\!\!\!\perp Y \,|\, Z)$ & $(X \perp\!\!\!\perp W \,|\, (Y, Z)) \Rightarrow$
$(X \perp\!\!\!\perp (W, Y) \,|\, Z)$

# Conclusion

- There are many different structural assessments for credal sets.
- Vacuity/uniformity/exchangeability are quite useful.
- Independence is the most important one.
- There are many different concepts of independence for credal sets.
- A study of (conditional) independence touches on
  - convexity and decision-making;
  - conditioning and full conditional measures.

# Final words on independence I

- Epistemic irrelevance/independence is quite intuitive and simple to state for convex credal sets.
  - Difficult to handle computationally.
  - Fails the contraction property (perhaps ok?).
  - Requires full conditional measures and associated challenges (perhaps then use type-5/regular independence?).

# Final words on independence II

- Complete independence is simple to state and inherits all the familiar properties of stochastic independence
  - Fails convexity, but this has behavioral meaning.
  - Nonlinear, but this is unavoidable in the end.
  - Can be adapted to full conditional measures (but need extra work).

# Final words on independence III

- Strong independence: popular because people want at once convexity and stochastic independence, no matter what.
  - It can be justified in some cases (exchangeability).
  - But hard to justify in general.