We assume that:

- There is only one queue
- There is only one server
- The capacity of the queueing system is 2
- There is maximally one arrival in one time step
- There is maximally one item serviced in one time step
- The service decision happens before the arrival event
The state of the system $X_k$ at time $k$ is an element of $\{z, o, t\}$ where $z$ corresponds to $X_k = 0$, $o$ to $X_k = 1$ and $t$ corresponds to $X_k = 2$. 

Unrolling the event tree
The relation between $X_k$, $A_k$ and $R_k$

The number of objects in the system $X_{k+1}$ (= in the buffer + being serviced) at time $k+1$, is determined by:

- $X_k$: The number of objects in the system at time $k$,
- $A_k$: The number of objects that have arrived at time $k$,
- $R_k$: The number of objects that have been serviced at time $k$,

$$X_{k+1} = X_k + A_k - R_k.$$ 

We assume that the only a limited number of combinations of $A_k$, $R_k$, $X_k$ and $X_{k+1}$ are allowed.

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Imprecise (stationary) Markov chain

- We assume that $A_k$ and $R_k$ do not depend on $A_{1:k-1}, R_{1:k-1}$.
- Consequently, $X_k$ is independent of $X_{1:k-1}$ which is the Markov condition and $\{X_k\}_{k \in \mathbb{N}}$ is a discrete time, imprecise Markov chain.
- As we furthermore assume that the belief model for $(A_k, R_k)$ does not depend on the time index $k$, the resulting Markov chain is stationary.

An imprecise stationary Markov chain is defined by

- its state space $\mathcal{X}$,
- the prior belief model $\overline{Q}_1$,
- the upper transition operator $\overline{T}$

\[
\overline{T}f(x) := \overline{Q}(f|x).
\]
Law of iterated expectation for Markov chains

- The advantage of interpreting the queueing system as an imprecise Markov chain is that any prevision (assuming epistemic irrelevance in the Markov condition) can be calculated recursively.
- When the gamble of interest depends on a single variable, then strong independence and epistemic irrelevance give the same result.

**Theorem**

For any real-valued map \( h \) on \( \mathcal{X}_n \), and for any \( 1 \leq \ell < n \) and all \( x_\ell \) in \( \mathcal{X}_\ell \):

\[
\bar{P}_{n|\ell}(h|x_\ell) = \bar{T}^{n-\ell}h(x_\ell),
\]

\[
\bar{P}_n(h) = \bar{Q}_1(\bar{T}^{n-1}h).
\]
What is the transition operator $\overline{T}$ equal to?

One of the difficulties in this problem is the calculation of the upper transition operator $\overline{T}$ itself. In order to calculate it, we assume that

- $A_k$ and $R_k$ are (as) independent (as possible).
- In this application, we choose strong independence.
- The belief models for $A_k$ and $R_k$ are parametrised binary belief models.
A small revision of binary belief models

For any $h \in \mathcal{L}_A$ we have that:

$\bar{P}_A(h) = \min \{ P(h) : P \in \mathcal{M}_A \}$

For any $g \in \mathcal{L}_R$ we have that:

$\bar{P}_R(g) = \min \{ P(g) : P \in \mathcal{M}_A \}$
A small revision of binary belief models

For any \( h \in \mathcal{L}_A \) we have that:

\[
\overline{P}_A(h) = \min \{ P(h) : P \in \mathcal{M}_A \}
\]
\[
\underline{P}_A(h) = \max \{ \beta : \beta - h \in \mathcal{D}_A \}
\]

For any \( g \in \mathcal{L}_R \) we have that:

\[
\overline{P}_R(g) = \min \{ P(g) : P \in \mathcal{M}_A \}
\]
\[
\underline{P}_R(g) = \max \{ \beta : \beta - g \in \mathcal{D}_A \}
\]
A small revision of binary belief models

For any $h \in \mathcal{L}_A$ we have that:

- $\overline{P}_A(h) = \min \{P(h) : P \in M_A\}$
- $\overline{P}_A(h) = \max \{\beta : \beta - h \in \mathcal{D}_A\}$
- $\overline{P}_A(h) = (1 - \varepsilon)A_0(h) + \varepsilon \max h$

For any $g \in \mathcal{L}_R$ we have that:

- $\overline{P}_R(g) = \min \{P(g) : P \in M_A\}$
- $\overline{P}_R(g) = \max \{\beta : \beta - g \in \mathcal{D}_A\}$
- $\overline{P}_R(g) = (1 - \varepsilon)R_0(g) + \varepsilon \max g$
The transition operator: $\overline{T}h(z)$

When $X_k = 0$, then $X_{k+1}$ is completely determined by the arrival process.

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Therefore, we have for any $h \in \mathcal{L}_{X_{k+1}}$, that

$$\overline{T}h(0) := \overline{Q}_{X_{k+1}}(h|X_k = 0)$$

$$= P_A(I_{A=0}h(z) + I_{A=1}h(o))$$

$$= (1 - \varepsilon)P_0(g) + \varepsilon \max g$$

where $g \in \mathcal{L}_{A_k}$ and $g(0) := h(z)$ and $g(1) := h(o)$. 
The transition operator: $\overline{Th}(o)$

We assume throughout the basis $B$ such that the matrix representation of the following indicators is given by

$$\begin{align*}
[I_{(A,R)=(0,0)}]_B &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & [I_{(A,R)=(0,1)}]_B &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
[I_{(A,R)=(1,0)}]_B &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & [I_{(A,R)=(1,1)}]_B &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}$$

The matrix representation of the extreme points of $\mathcal{M}_A$ and $\mathcal{M}_R$ are then given by

$$\begin{align*}
[\text{ext } \mathcal{M}_A]_B &= \{ (1 - \bar{a} \ 1 - \bar{a} \ \bar{a} \ \bar{a} \ \bar{a}) , (1 - a \ 1 - a \ a \ a) \} , \\
[\text{ext } \mathcal{M}_R]_B &= \{ (1 - \bar{r} \ \bar{r} \ 1 - \bar{r} \ \bar{r}) , (1 - r \ r \ 1 - r \ r) \}
\end{align*}$$
If we assume that $A$ and $R$ are strongly independent, then the extreme points of their strong product $\mathcal{M}_A \boxtimes \mathcal{M}_R$ are given by every combination of product of extreme points of $\mathcal{M}_A$ and $\mathcal{M}_R$:

$$\text{ext}(\mathcal{M}_A \boxtimes \mathcal{M}_R) = \{s t : s \in \text{ext}(\mathcal{M}_A) \text{ and } t \in \text{ext}(\mathcal{M}_A)\}$$

whence

$$[\text{ext}(\mathcal{M}_A \boxtimes \mathcal{M}_R)]_B = \left\{ \begin{array}{l}
((1 - \bar{a})(1 - \bar{r}) \quad (1 - \bar{a})\bar{r} \quad \bar{a}(1 - \bar{r}) \quad \bar{a}\bar{r}) , \\
((1 - \bar{a})(1 - r) \quad (1 - \bar{a})r \quad \bar{a}(1 - r) \quad \bar{a}r) , \\
((1 - a)(1 - \bar{r}) \quad (1 - a)\bar{r} \quad a(1 - \bar{r}) \quad ar) , \\
((1 - a)(1 - r) \quad (1 - a)r \quad a(1 - r) \quad ar) \end{array} \right\}$$
\( \overline{T}h(o) \)

- We are not interested in gambles on \( \mathcal{A} \times \mathcal{R} = \{0, 1\} \times \{0, 1\} \), but in gambles on \( \mathcal{X} = \{z, o, t\} \) (coarsening).
- If \( X_k = o \), then it easy to see that

\[
\begin{align*}
I_{X_{k+1}} = z &= I(A, R) = (0, 1), \\
I_{X_{k+1}} = o &= I(A, R) = (0, 0) + I(A, R) = (1, 1), \\
I_{X_{k+1}} = t &= I(A, R) = (1, 0).
\end{align*}
\]

- Therefore, we see that for any \( h \in \mathcal{L}_{X_{k+1}} \)

\[
\overline{T}h(o) := \overline{Q}_{X_{k+1}}(h|X_k = o) \\
= P_{A \boxtimes R}(I(A, R) = (0, 1) h(z) + I(A, R) \in \{(0, 0), (1, 1)\} h(o) + I(A, R) = (1, 0) h(t)) \\
= \min \left\{ \begin{pmatrix} (1 - a)\bar{r} \\ ar + (1 - a)(1 - r) \\ a(1 - r) \end{pmatrix}^T \begin{pmatrix} h(z) \\ h(o) \\ h(t) \end{pmatrix} : a \in \{a, \bar{a}\}, r \in \{r, \bar{r}\} \right\}. 
\]
The extreme transition matrices

In a similar fashion, we get for any \( h \in L_{X_k} \) that

\[
\mathbf{T} h(t) = \min \left\{ \left( \begin{array}{c}
(1-a)r \\
a + (1-a)(1-r)
\end{array} \right)^T \cdot \left( \begin{array}{c}
h(0) \\
 h(z)
\end{array} \right) : a \in \{a, \bar{a}\}, r \in \{r, \bar{r}\} \right\}
\]

We can summarise our findings about \( \mathbf{T} \) in the extreme transition matrices (16 in total)

\[
\begin{pmatrix}
(1-a_0) & a_0 & 0 \\
(1-a_1)r_1 & (1-a_1)(1-r_1) + a_1r_1 & a_1(1-r_1) \\
0 & (1-a_2)r_2 & a_2 + (1-a_2)(1-r_2)
\end{pmatrix}
\]

where \( a_0, a_1, a_2 \in \{a, \bar{a}\} \) and \( r_0, r_1, r_2 \in \{r, \bar{r}\} \).
The transition operator on the simplex

\[
\begin{align*}
a_0 &= \frac{2}{10} & r_0 &= \frac{2}{10} & \epsilon &= \frac{300}{1000} \\
\epsilon &= \frac{300}{1000} & \\
\end{align*}
\]

\[
\begin{align*}
a_0 &= \frac{4}{10} & r_0 &= \frac{6}{10} & \epsilon &= \frac{300}{1000} \\
\epsilon &= \frac{300}{1000} & \\
\end{align*}
\]

\[
\begin{align*}
a_0 &= \frac{8}{10} & r_0 &= \frac{2}{10} & \epsilon &= \frac{300}{1000} \\
\epsilon &= \frac{300}{1000} & \\
\end{align*}
\]

\[
\begin{align*}
a_0 &= \frac{2}{10} & r_0 &= \frac{4}{10} & \epsilon &= \frac{10}{1000} \\
\epsilon &= \frac{10}{1000} & \\
\end{align*}
\]

\[
\begin{align*}
a_0 &= \frac{2}{10} & r_0 &= \frac{4}{10} & \epsilon &= \frac{100}{1000} \\
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\]

\[
\begin{align*}
a_0 &= \frac{2}{10} & r_0 &= \frac{4}{10} & \epsilon &= \frac{1000}{1000} \\
\epsilon &= \frac{1000}{1000} & \\
\end{align*}
\]
Time evolution and ergodicity

\[ a_0 = \frac{7}{10} \quad \text{and} \quad r_0 = \frac{4}{10} \quad \text{and} \quad \varepsilon = \frac{2}{10}. \]
Influence of the precise models

\[ a_0 = \frac{4}{10} \quad r_0 = \frac{0}{10} \]

\[ a_0 = \frac{4}{10} \quad r_0 = \frac{2}{10} \]

\[ a_0 = \frac{4}{10} \quad r_0 = \frac{4}{10} \]

\[ a_0 = \frac{4}{10} \quad r_0 = \frac{6}{10} \]

\[ a_0 = \frac{4}{10} \quad r_0 = \frac{8}{10} \]

\[ a_0 = \frac{4}{10} \quad r_0 = \frac{10}{10} \]

\[ a_0 = \frac{8}{10} \quad r_0 = \frac{0}{10} \]

\[ a_0 = \frac{2}{10} \quad r_0 = \frac{10}{10} \]

\[ n = 5000 \quad \text{and} \quad \varepsilon = \frac{1}{10}. \]
Influence of imprecision

\[ \varepsilon = 0.0000 \quad \varepsilon = 0.0100 \quad \varepsilon = 0.0200 \quad \varepsilon = 0.0500 \]

\[ \varepsilon = 0.1000 \quad \varepsilon = 0.2000 \quad \varepsilon = 0.5000 \quad \varepsilon = 1.0000 \]

\[ n = 5000 \text{ and } a_0 = \frac{7}{10} \text{ and } r_0 = \frac{4}{10}. \]
Consider a stationary imprecise Markov chain with finite state set $\mathcal{X}$ that is ergodic. Then for every initial upper expectation $\overline{P}_1$, the upper expectation $\overline{P}_n = \overline{P}_1 \circ \overline{T}^{n-1}$ for the state at time $n$ converges point-wise to the same upper expectation $\overline{P}_\infty$:

$$\lim_{n \to \infty} \overline{P}_n(h) = \lim_{n \to \infty} \overline{P}_1(\overline{T}^{n-1}h) =: \overline{P}_\infty(h) \text{ for all } h \text{ in } \mathcal{L}[\mathcal{X}].$$

Moreover, the limit upper expectation $\overline{P}_\infty$ is the only $\overline{T}$-invariant upper expectation on $\mathcal{L}[\mathcal{X}]$.

In practice, checking for ergodicity results in checking

1. Top class regularity
2. Top class absorption
How to check for top class regularity?

- Draw a directed graph \((\mathcal{X}, E)\) where \((x, y) \in E \iff \overline{TI}_{\{y\}}(x) > 0\).

\[
gcd N_{tt} = \gcd N = 1 \Rightarrow \text{aperiodic top class}
\]

- Check whether there is a top class \(\mathcal{R} \subseteq \mathcal{X}^\ast\), i.e. a unique maximal communication class, i.e. a unique strongly connected component of the graph that has no outgoing edges.

- Check whether this top class is aperiodic

\[
gcd N_{aa} = \gcd \{\alpha 6 + \beta 8 : \alpha \in \mathbb{N}, \beta \in \mathbb{N}\} = 2.
\]

Only paths from \(a\) to \(a\) exists that have a length that is a multiple of 2 \(\Rightarrow\) periodic.
Checking for top class absorption

Check whether it is guaranteed that, eventually, the Markov chain evolves from any situation to a situation in the top class.

\[(\forall z \notin \mathcal{R})(\exists n \in \mathbb{N})(\bar{T}^n\mathcal{R}(z) > 0)\]

Theorem (Top class absorption)
Let \(\bar{T}\) be an upper transition operator with regular top class \(\mathcal{R}\). Consider the nested sequence of subsets of \(\mathcal{R}^c\) defined by the iterative scheme:

\[A_0 := \mathcal{R}^c \quad \text{and} \quad A_{n+1} := \{a \in A_n : \bar{T}I_{A_n}(a) = 1\}, \quad n \geq 0.\]

After \(k \leq |\mathcal{R}^c|\) iterations, we reach \(A_k = A_{k+1}\). Then \(\bar{T}\) is top class absorbing if and only if \(A_k = \emptyset\).
Example – top class regularity

Define $\overline{Tf} = \max \{Mf : L \leq M \leq U \text{ and } M \text{ stochastic}\}$ where $L$ and $U$ are given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/4 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 3/4 & 1/2 & 0 & 0 \\ 3/4 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1/4 & 3/4 & 0 & 0 & 1/4 \end{pmatrix}.$$  

The corresponding upper accessibility graph is given by

{1} corresponds to the unique strongly connected component that is final. As it is a singleton, it has cyclicity one, so there is a regular top class $\mathcal{R} = \{1\}$. 
Example - top class absorption

\[ L = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{4}
\end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{3}{4} & \frac{1}{2} & 0 & 0 \\
\frac{3}{4} & \frac{1}{2} & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 & \frac{1}{4}
\end{pmatrix}.\]

We know that \( R = \{1\} \). To check for top class absorption, we start iterating:

1. \( \overline{T}I_{R^c} = (0 \ 1 \ \frac{1}{2} \ 1 \ 1)^T \Rightarrow I_{A_1} = (0 \ 1 \ 0 \ 1 \ 1)^T, \)
2. \( \overline{T}I_{A_1} = (0 \ \frac{3}{4} \ \frac{1}{2} \ 1 \ 1)^T \Rightarrow I_{A_2} = (0 \ 0 \ 0 \ 1 \ 1)^T, \)
3. \( \overline{T}I_{A_2} = (0 \ 0 \ 0 \ 1 \ \frac{1}{4})^T \Rightarrow I_{A_3} = (0 \ 0 \ 0 \ 1 \ 0)^T, \)
4. \( \overline{T}I_{A_3} = (0 \ 0 \ 0 \ 1 \ 0)^T \Rightarrow I_{A_4} = (0 \ 0 \ 0 \ 1 \ 0)^T. \)

Because \( I_{A_4} = I_{A_3} \neq 0 \) we conclude that \( \overline{T} \) is not top class absorbing and therefore not ergodic.