CREDAL TREES
UNDER EPISTEMIC IRRELEVANCE

Gert de Cooman Jasper De Bock

Ghent University, SYSTeMS

{gert.decooman,jasper.debock}@UGent.be

5th SIPTA School
19 July 2012

What would we like to achieve and convey?

What would we like to achieve and convey?

CREDAL TREES UNDER IRRELEVANCE

interpretation of the graphical structure

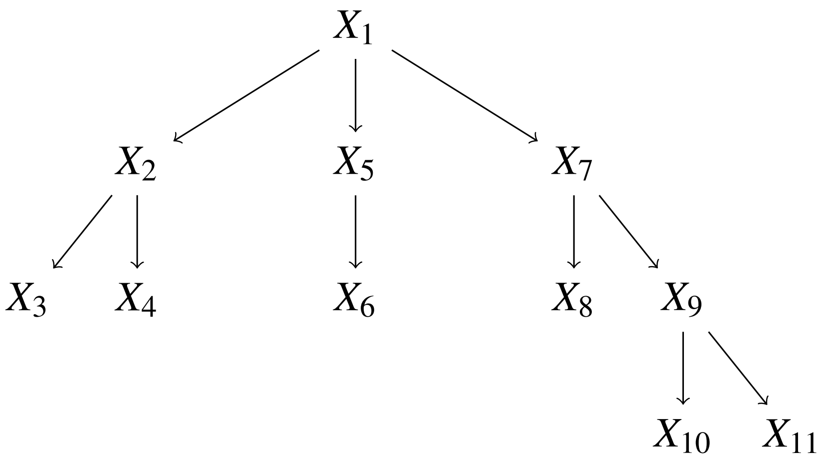
recursive construction of the joint

Credal trees under epistemic irrelevance

Credal networks: the special case of a tree

Basic concept

Consider a directed tree T, with a variable X_t attached to each node $t \in T$.



Each variable X_t assumes values in a set $\mathscr{X}_t$.

Credal trees: local uncertainty models

Local uncertainty model associated with each node t

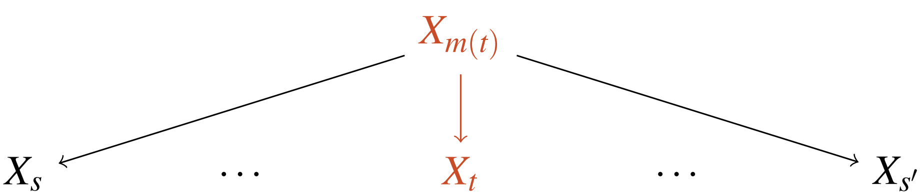
For each possible value $x_{m(t)} \in \mathscr{X}_{m(t)}$ of the mother variable $X_{m(t)}$, we have a conditional lower prevision/expectation

$$\underline{Q}_t(\cdot|x_{m(t)}) \colon \mathscr{L}(\mathscr{X}_t) \to \mathbb{R}$$

where

$$\underline{Q}_t(f|x_{m(t)}) = \text{lower prevision of } f(X_t) \text{, given that } X_{m(t)} = x_{m(t)}.$$

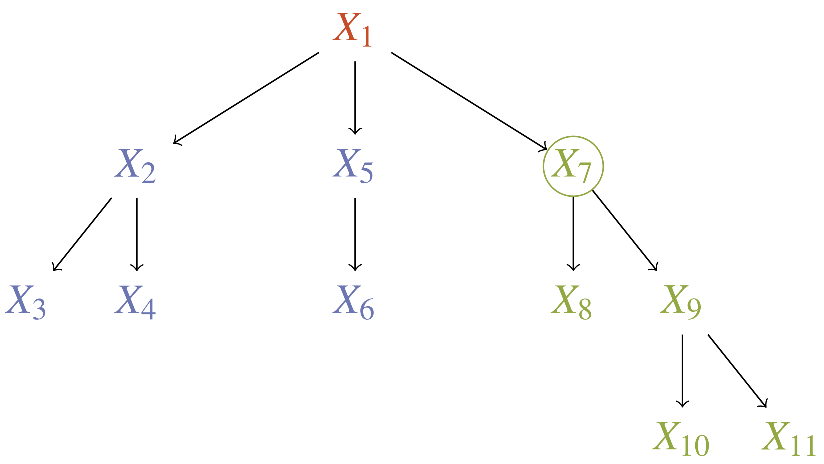
The local model $\underline{Q}_t(\cdot|X_{m(t)})$ is a conditional lower prevision operator.



Interpretation of the graphical structure

The graphical structure is interpreted as follows:

Conditional on the mother variable, the non-parent non-descendants of each node variable are *epistemically irrelevant* to it and its descendants.



Constructing the joint recursively

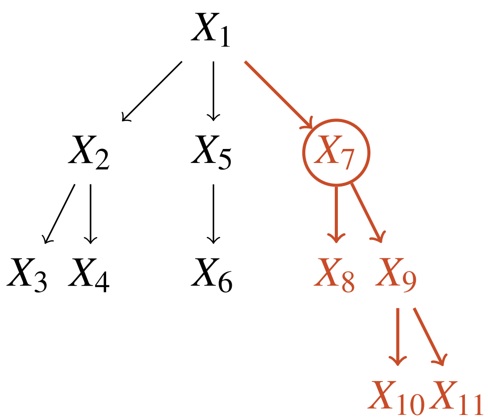
Constructing global from local models

For each node t, we want to construct a global model:

$$\underline{P}_t(\cdot|X_{m(t)})\colon \mathscr{L}(\mathscr{X}_{\downarrow t}) \to \mathbb{R}$$

representing the uncertainty about all variables $X_{\downarrow t}$ in the subtree $\downarrow t$ below t:

$$\downarrow t = \{t\} \cup \{s\colon s \text{ is a descendant of } t\}$$



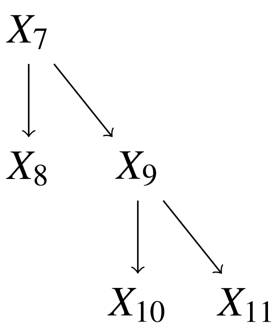
Constructing global from local models

For each node t, we want to construct a global model:

$$\underline{P}_t(\cdot|X_{m(t)})\colon \mathscr{L}(\mathscr{X}_{\downarrow t}) \to \mathbb{R}$$

representing the uncertainty about all variables $X_{\downarrow t}$ in the subtree $\downarrow t$ below t:

$$\downarrow t = \{t\} \cup \{s\colon s \text{ is a descendant of } t\}$$



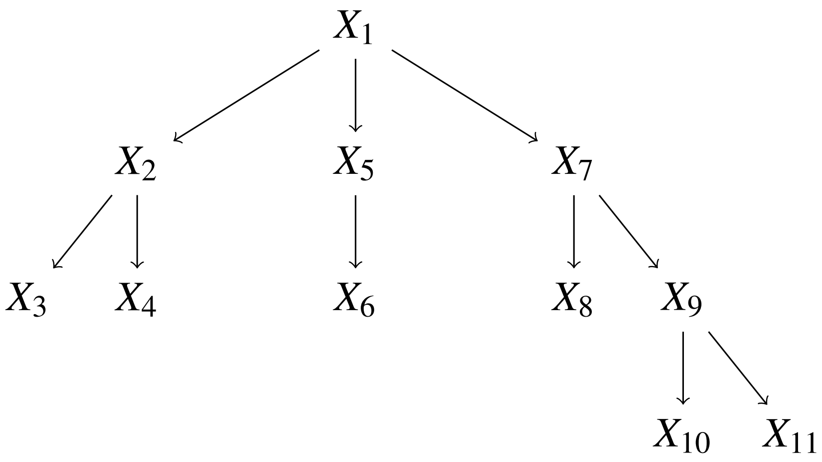
$$\downarrow 7 = \{7, 8, 9, 10, 11\}$$

$$X_{\downarrow 7} = (X_7, X_8, X_9, X_{10}, X_{11})$$

$$X_{m(7)} = X_1$$

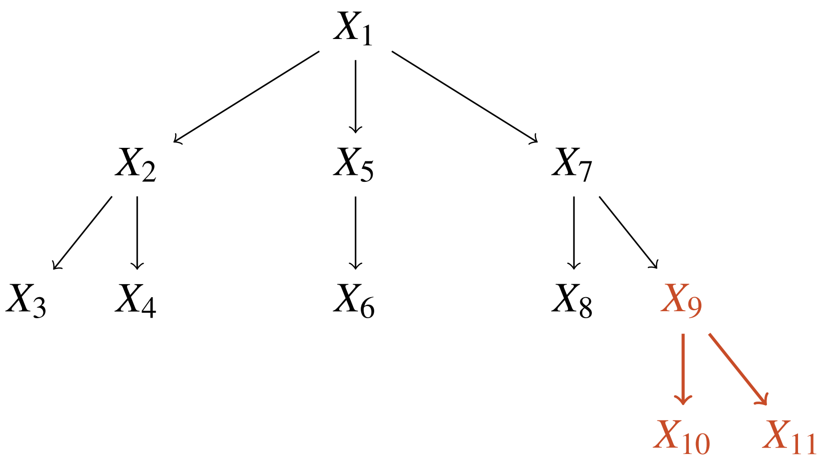
First crucial observation

We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.



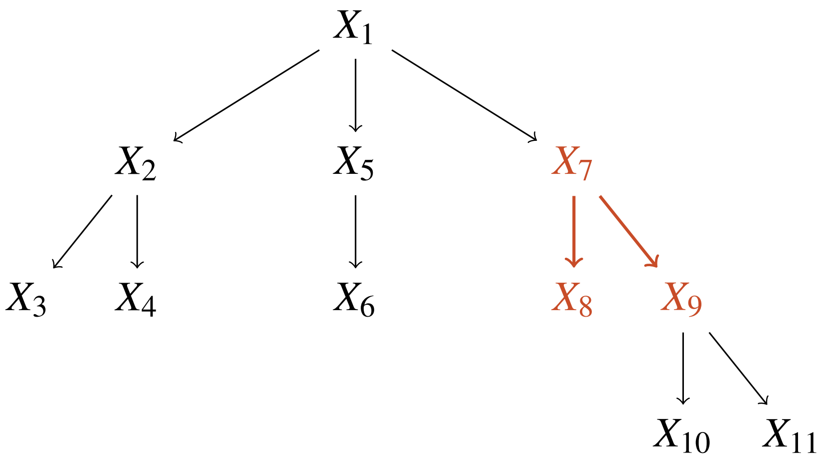
First crucial observation

We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.



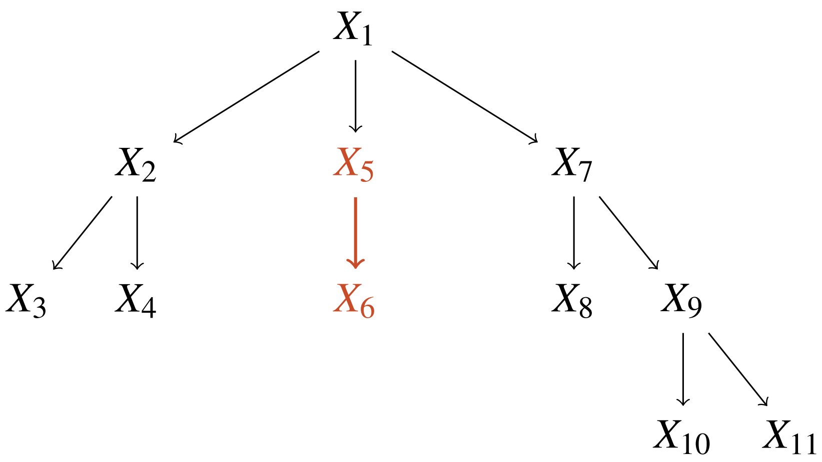
First crucial observation

We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.



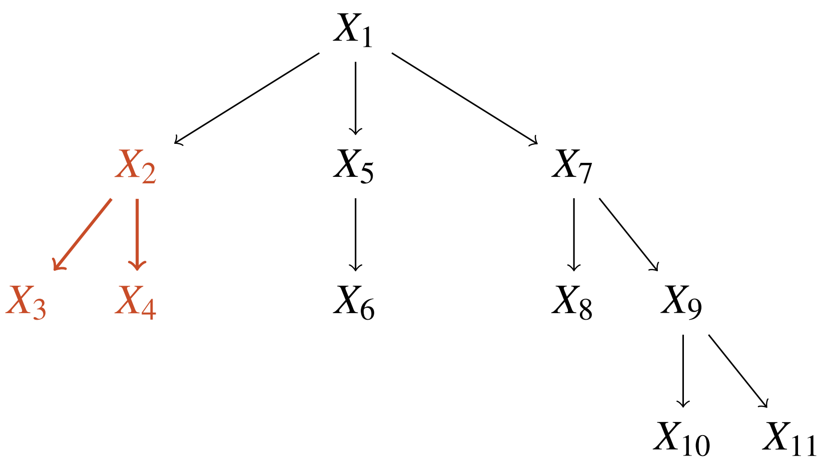
First crucial observation

We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.



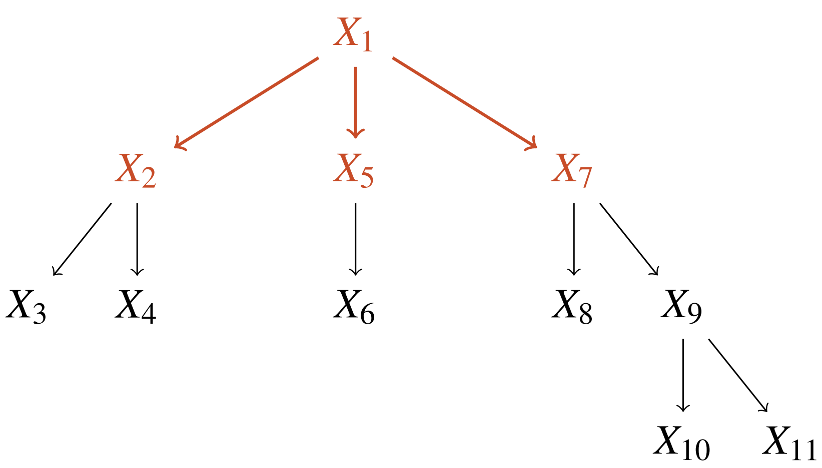
First crucial observation

We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.

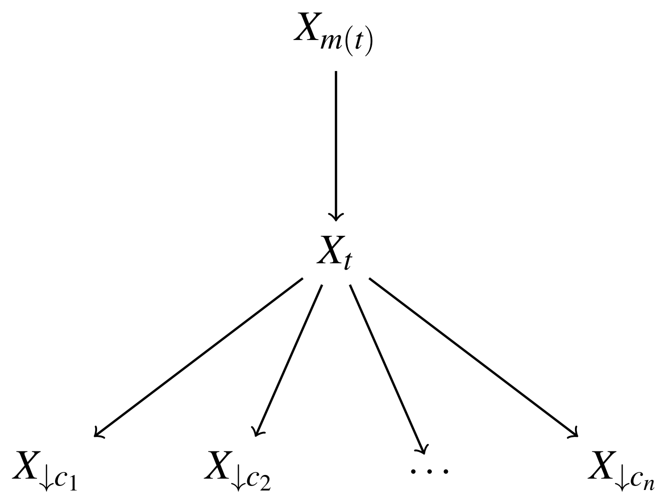


First crucial observation

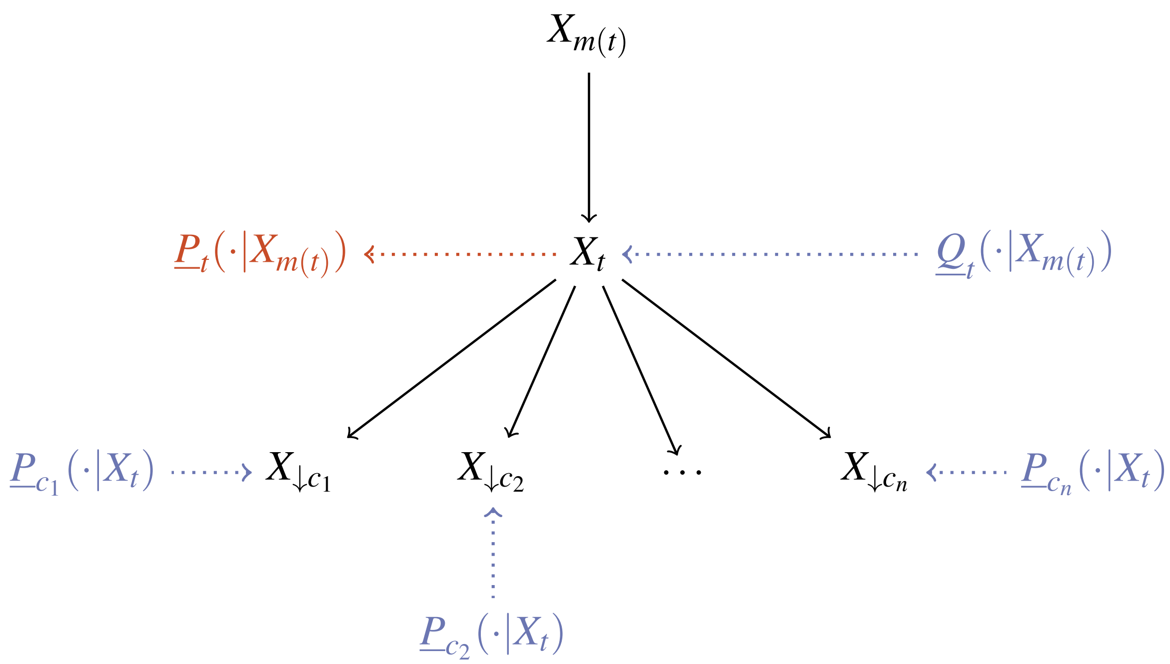
We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.



Graphical representation of the recursion

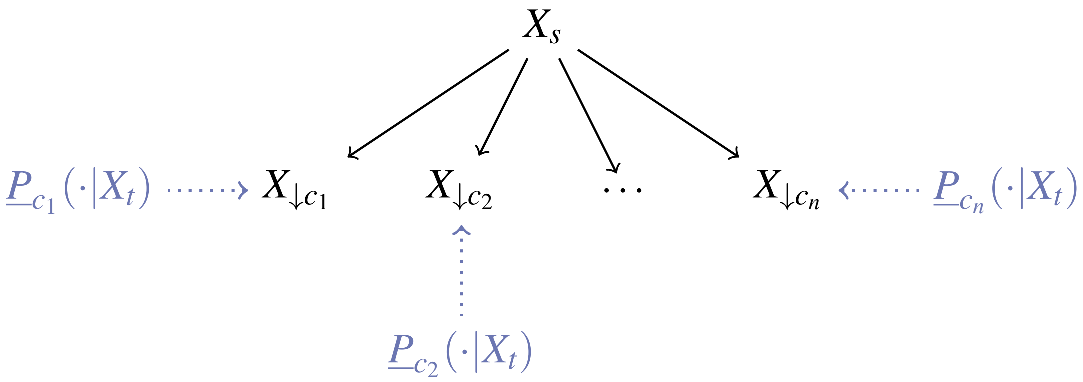


Graphical representation of the recursion



First step: independent natural extension

Conditional on X_t, the children X_c, $c \in \mathrm{chld}(t)$ are epistemically independent.

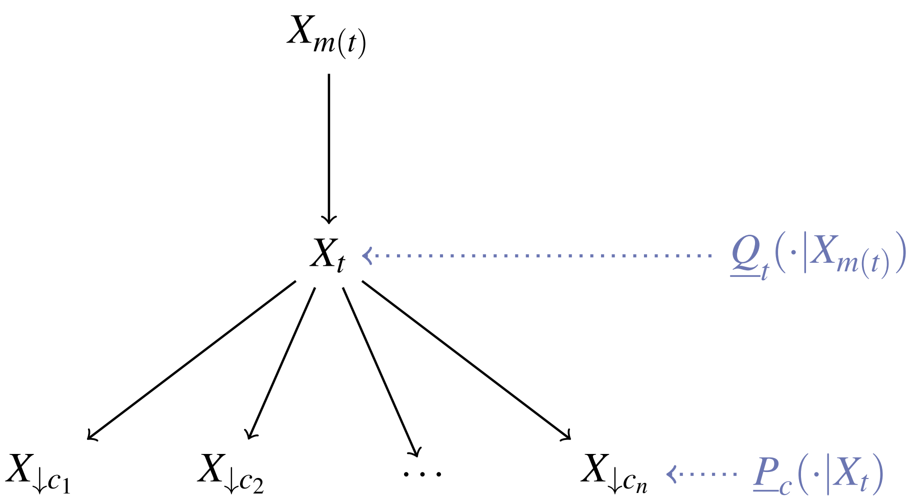


Independent natural extension $\underline{E}_t(\cdot|X_t)$ on $\mathscr{L}(\mathscr{X}_{\downarrow t \setminus \{t\}})$:

$$\underline{E}_t(\cdot|x_t) = \otimes_{c \in \mathrm{chld}(t)} \underline{P}_c(\cdot|x_t) \text{ for all } x_t \in \mathscr{X}_t$$

First step: independent natural extension

Conditional on X_t, the children X_c, $c \in \mathrm{chld}(t)$ are epistemically independent.

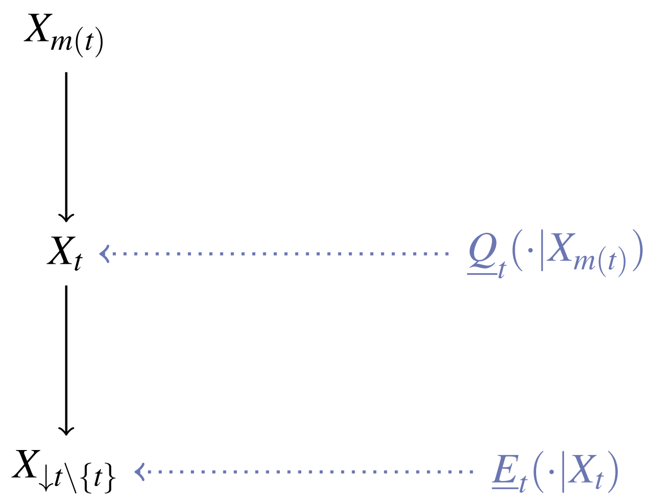


Independent natural extension $\underline{E}_t(\cdot|X_t)$ on $\mathscr{L}(\mathscr{X}_{\downarrow t \setminus \{t\}})$:

$$\underline{E}_t(\cdot|x_t) = \otimes_{c \in \mathrm{chld}(t)} \underline{P}_c(\cdot|x_t) \text{ for all } x_t \in \mathscr{X}_t$$

First step: independent natural extension

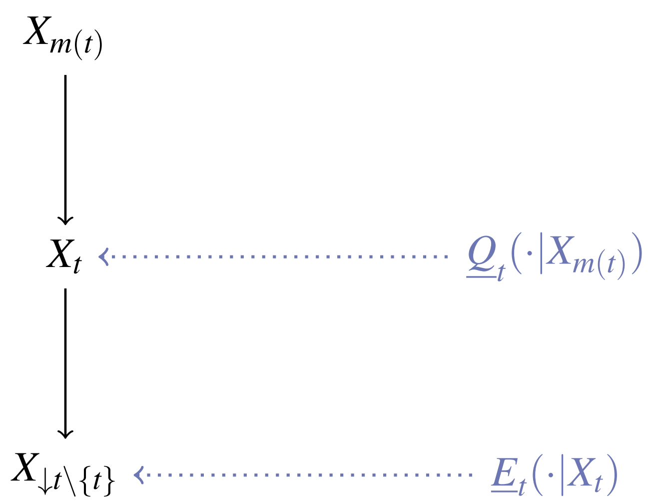
Conditional on X_t, the children X_c, $c \in \mathrm{chld}(t)$ are epistemically independent.



Independent natural extension $\underline{E}_t(\cdot|X_t)$ on $\mathscr{L}(\mathscr{X}_{\downarrow t\setminus\{t\}})$:

$$\underline{E}_t(\cdot|x_t) = \otimes_{c\in\mathrm{chld}(t)}\underline{P}_c(\cdot|x_t) \text{ for all } x_t \in \mathscr{X}_t$$

Step 2: marginal extension



Step 2: marginal extension

Bayes's Rule:

$$p(x_2, x_3|x_1) = p(x_3|x_2, x_1)p(x_2|x_1) = p(x_3|x_2)p(x_2|x_1)$$

Taking expectations:

$$\begin{aligned}
\sum_{\substack{x_2 \in \mathcal{X}_2 \\ x_3 \in \mathcal{X}_3}} f(x_2, x_3)p(x_2, x_3|x_1) &= \sum_{\substack{x_2 \in \mathcal{X}_2 \\ x_3 \in \mathcal{X}_3}} f(x_2, x_3)p(x_3|x_2)p(x_2|x_1) \\
&= \sum_{x_2 \in \mathcal{X}_2} p(x_2|x_1) \sum_{x_3 \in \mathcal{X}_3} f(x_2, x_3)p(x_3|x_2) \\
&= \sum_{x_2 \in \mathcal{X}_2} p(x_2|x_1)E(f(x_2, X_3)|x_2) \\
&= E\left(E(f(X_2, X_3)|X_2)|x_1\right)
\end{aligned}$$

Law of iterated expectations:

$$E(f(X_2, X_3)|X_1) = E\left(E(f(X_2, X_3)|X_2)|X_1\right)$$

Step 2: marginal extension

Bayes's Rule:

$$p(x_2, x_3|x_1) = p(x_3|x_2, x_1)p(x_2|x_1) = p(x_3|x_2)p(x_2|x_1)$$

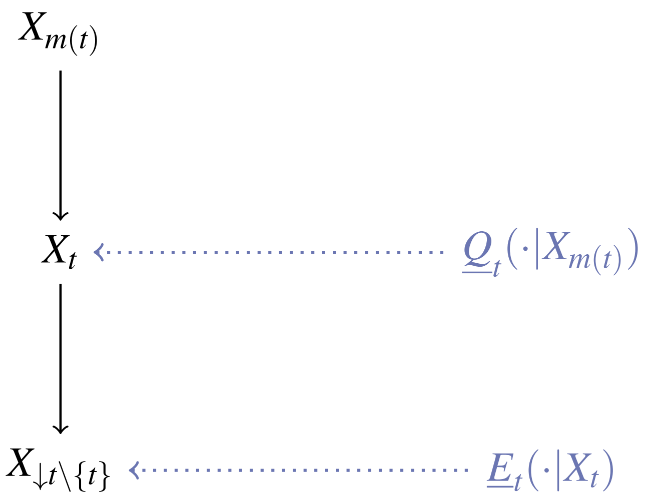
Taking expectations:

$$\begin{aligned}
\sum_{\substack{x_2 \in \mathcal{X}_2 \\ x_3 \in \mathcal{X}_3}} f(x_2, x_3)p(x_2, x_3|x_1) &= \sum_{\substack{x_2 \in \mathcal{X}_2 \\ x_3 \in \mathcal{X}_3}} f(x_2, x_3)p(x_3|x_2)p(x_2|x_1) \\
&= \sum_{x_2 \in \mathcal{X}_2} p(x_2|x_1) \sum_{x_3 \in \mathcal{X}_3} f(x_2, x_3)p(x_3|x_2) \\
&= \sum_{x_2 \in \mathcal{X}_2} p(x_2|x_1)E(f(x_2, X_3)|x_2) \\
&= E\left(E(f(X_2, X_3)|X_2)|x_1\right)
\end{aligned}$$

Law of iterated expectations:

$$\underline{P}(f(X_2, X_3)|X_1) = \underline{P}\left(\underline{P}(f(X_2, X_3)|X_2)|X_1\right)$$

Step 2: marginal extension



Marginal extension (or law of iterated expectations):

$$\underline{P}_t(\cdot|X_{m(t)}) = \underline{Q}_t(\underline{E}_t(\cdot|X_t)|X_{m(t)}) = \underline{Q}_t(\otimes_{c\in\mathrm{chld}(t)}\underline{P}_c(\cdot|X_t)|X_{m(t)})$$

Step 2: marginal extension

Marginal extension (or law of iterated expectations):

$$\underline{P}_t(\cdot|X_{m(t)}) = \underline{Q}_t(\underline{E}_t(\cdot|X_t)|X_{m(t)}) = \underline{Q}_t(\otimes_{c\in\mathrm{chld}(t)}\underline{P}_c(\cdot|X_t)|X_{m(t)})$$

Recursive construction of the joint: summary

We start at the leaves s with:

$$\underline{P}_s(\cdot|X_{m(s)}) := \underline{Q}_s(\cdot|X_{m(s)}) \text{ for all leaves } s$$

and move recursively upwards to the root of the tree:

$$\left.\begin{aligned} \underline{E}_t(\cdot|X_t) &:= \otimes_{c\in\mathrm{chld}(t)}\underline{P}_c(\cdot|X_t) \\ \underline{P}_t(\cdot|X_{m(t)}) &:= \underline{Q}_t(\underline{E}_t(\cdot|X_t)|X_{m(t)}) \end{aligned}\right\} \text{ for all non-terminal nodes } t.$$

Fundamental result

Theorem

In this recursive fashion, we construct the point-wise smallest (most conservative) collection of global models $\underline{P}_s(\cdot|X_{m(s)})$ that

(i) is coherent;

(ii) is compatible with the local information;

(iii) reflects all the epistemic irrelevancies embodied in the graphical structure.

Important remark:

This recursive construction of the joint is only valid if we impose (the asymmetrical) irrelevance, not if we impose (the symmetrical) independence: in the latter case the development is more complicated and computationally less efficient.

What would we like to achieve and convey?

CREDAL TREES UNDER IRRELEVANCE	MePiCTIr	
interpretation of the graphical structure	essential ideas	
recursive construction of the joint	example	

MePiCTIr algorithm

Credal networks under epistemic irrelevance

For a credal tree we can find the joint model from the local models recursively, from leaves to root.

Exact message passing algorithm

- credal tree treated as an expert system
- linear complexity in the number of nodes

Python code

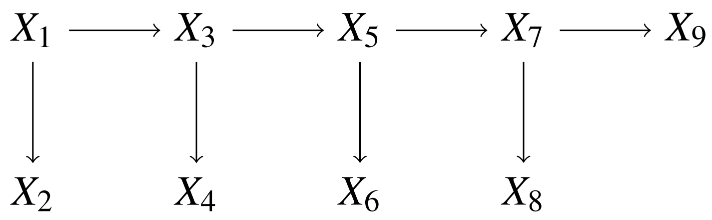
- written by Filip Hermans
- testing and connection with strong independence in cooperation with Marco Zaffalon and Alessandro Antonucci

Current (toy) applications in HMMs

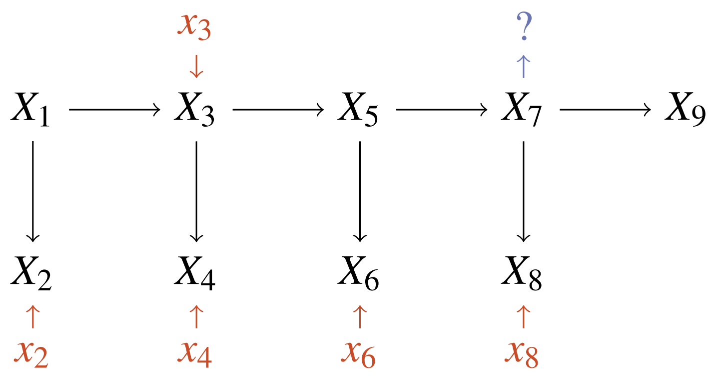
character recognition, air traffic trajectory tracking and identification, earthquake rate prediction

Explaining the basics of the algorithm

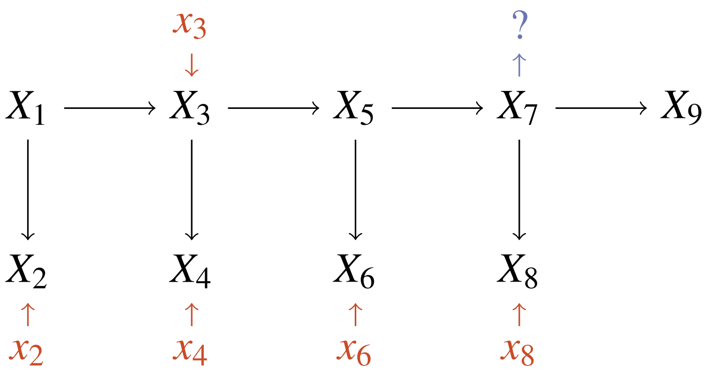
Basics of the algorithm: an example



Basics of the algorithm: an example



Basics of the algorithm: an example



Regular extension

$$\underline{R}(g(X_7)|X_2 = x_2, X_3 = x_3, X_4 = x_4, X_6 = x_6, X_8 = x_8)$$
$$= \max\left\{\mu \in \mathbb{R} \colon \underline{P}_1\left(\mathbb{I}_{\{x_2\}}\mathbb{I}_{\{x_3\}}\mathbb{I}_{\{x_4\}}\mathbb{I}_{\{x_6\}}\mathbb{I}_{\{x_8\}}[g(X_7) - \mu]\right)\right\}$$

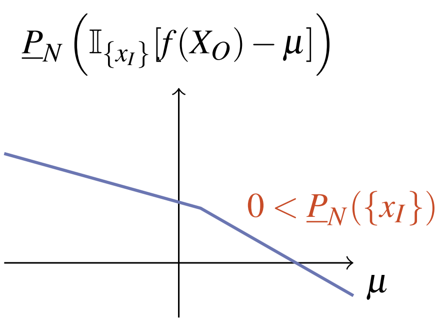
Intermezzo: regular extension

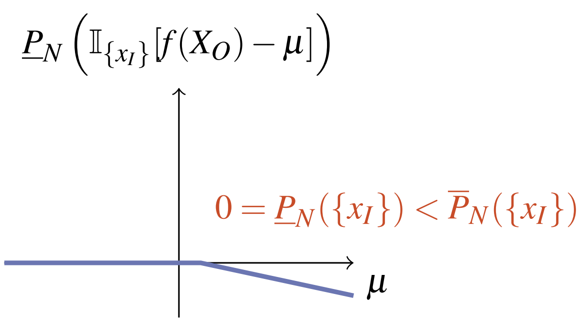
Recall:

$$\underline{P}_I(g(X_O)|x_O) = \sup\left\{\mu \in \mathbb{R} \colon \mathbb{I}_{\{x_I\}}[g(X_O) - \mu] \in \mathscr{D}_N\right\}$$

But we only know the joint lower prevision $\underline{P}_N$, not the set $\mathscr{D}_N$:
Regular extension:

$$\underline{R}(g(X_O)|x_I) := \sup\left\{\mu \in \mathbb{R} \colon \mathbb{I}_{\{x_I\}}[f(X_O) - \mu] \in \mathrm{cl}(\mathscr{D}_N)\right\}$$
$$:= \max\left\{\mu \in \mathbb{R} \colon \underline{P}_N\left(\mathbb{I}_{\{x_I\}}[f(X_O) - \mu]\right) \geq 0\right\}$$





Basics of the algorithm: calculating the joint

$$\begin{array}{ccccccccc} & & \overset{x_3}{\check{}} & & & & \overset{?}{\hat{}} & & \\ X_1 & \to & X_3 & \to & X_5 & \to & X_7 & \to & X_9 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X_2 & & X_4 & & X_6 & & X_8 & & \\ \hat{x}_2 & & \hat{x}_4 & & \hat{x}_6 & & \hat{x}_8 & & \end{array}$$

We have to calculate $\underline{P}_1(f_1)$, where

$$f_1 = \mathbb{I}_{\{x_2\}}\mathbb{I}_{\{x_3\}}\mathbb{I}_{\{x_4\}}\mathbb{I}_{\{x_6\}}\mathbb{I}_{\{x_8\}}[g(X_7) - \mu] = \mathbb{I}_{\{x_2\}} f_3$$

Now use the recursion formula for $\underline{P}_1$:

$$\underline{P}_1(f_1) = \underline{Q}_1(\underline{E}_1(f_1|X_1))$$

and the factorisation property of independent natural extension:

$$\underline{E}_1(f_1|X_1) = \underline{E}_1(\mathbb{I}_{\{x_2\}} f_3|X_1) = \overline{Q}_2(\{x_2\}|X_1) \odot \underline{P}_3(f_3|X_1)$$

Basics of the algorithm: one level down

$$\begin{array}{ccccccccc} & & \overset{x_3}{\check{}} & & & & \overset{?}{\hat{}} & & \\ X_1 & \to & X_3 & \to & X_5 & \to & X_7 & \to & X_9 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X_2 & & X_4 & & X_6 & & X_8 & & \\ \hat{x}_2 & & \hat{x}_4 & & \hat{x}_6 & & \hat{x}_8 & & \end{array}$$

We have to calculate $\underline{P}_3(f_3|X_1)$, where

$$f_3 = \mathbb{I}_{\{x_3\}}\mathbb{I}_{\{x_4\}}\mathbb{I}_{\{x_6\}}\mathbb{I}_{\{x_8\}}[g(X_7) - \mu] = \mathbb{I}_{\{x_3\}}\mathbb{I}_{\{x_4\}} f_5$$

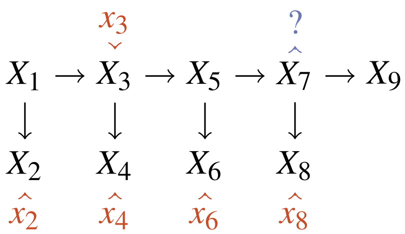
Now use the recursion formula for $\underline{P}_3(\cdot|X_1)$:

$$\underline{P}_3(f_3|X_1) = \underline{Q}_3(\underline{E}_3(f_3|X_3)|X_1)$$

and the factorisation property of independent natural extension:

$$\underline{E}_3(f_3|X_3) = \underline{E}_3(\mathbb{I}_{\{x_3\}}\mathbb{I}_{\{x_4\}} f_5|X_3) = \mathbb{I}_{\{x_3\}}\overline{Q}_4(\{x_4\}|x_3) \odot \underline{P}_5(f_5|x_3)$$

Basics of the algorithm: summary so far



In summary, so far we have:

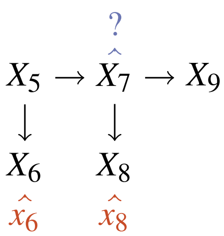
$$\underline{P}_3(f_3|X_1) = \overline{\underline{Q}}_3(\{x_3\}|X_1) \odot \overline{\underline{Q}}_4(\{x_4\}|x_3) \odot \underline{P}_5(f_5|x_3)$$

$$\underline{E}_1(f_1|X_1) = \overline{\underline{Q}}_2(\{x_2\}|X_1) \odot \overline{\underline{Q}}_3(\{x_3\}|X_1) \odot \overline{\underline{Q}}_4(\{x_4\}|x_3) \odot \underline{P}_5(f_5|x_3)$$

$$\underline{P}_1(f_1) = \overline{\underline{Q}}_1\left(\overline{\underline{Q}}_2(\{x_2\}|X_1)\overline{\underline{Q}}_3(\{x_3\}|X_1)\right)\overline{\underline{Q}}_4(\{x_4\}|x_3) \odot \underline{P}_5(f_5|x_3)$$

$$= \overline{\underline{P}}_1(\{(x_2,x_3,x_4)\}) \odot \underline{P}_5(f_5|x_3)$$

Basics of the algorithm: pruning the irrelevant part



Because $\underline{P}_1(f_1) \geq 0 \Leftrightarrow \underline{P}_5(f_5|x_3) \geq 0$, we have to calculate $\underline{P}_5(f_5|x_3)$, where

$$f_5 = \mathbb{I}_{\{x_6\}}\mathbb{I}_{\{x_8\}}[g(X_7)-\mu] = \mathbb{I}_{\{x_6\}}f_7$$

Now use the recursion formula for $\underline{P}_5(\cdot|x_3)$:

$$\underline{P}_5(f_5|x_3) = \underline{Q}_5(\underline{E}_5(f_5|X_5)|x_3)$$

and the factorisation property of independent natural extension:

$$\underline{E}_5(f_5|X_5) = \underline{E}_5(\mathbb{I}_{\{x_6\}}f_7|X_5) = \overline{\underline{Q}}_6(\{x_6\}|X_5) \odot \underline{P}_7(f_7|X_5)$$

Basics of the algorithm: one level down

$$
\begin{array}{ccccc}
 & & ? & & \\
 & & \hat{} & & \\
X_5 & \rightarrow & X_7 & \rightarrow & X_9 \\
\downarrow & & \downarrow & & \\
X_6 & & X_8 & & \\
\hat{x}_6 & & \hat{x}_8 & &
\end{array}
$$

We have to calculate $\underline{P}_7(f_7|X_5)$, where

$$f_7 = \mathbb{I}_{\{x_8\}}[g(X_7) - \mu]$$

Now use the recursion formula for $\underline{P}_7(\cdot|X_5)$:

$$\underline{P}_7(f_7|X_5) = \underline{Q}_7(\underline{E}_7(f_7|X_7)|X_5)$$

and apply independent natural extension:

$$\underline{E}_7(\mathbb{I}_{\{x_8\}}[g(X_7) - \mu]|X_7) = \overline{\underline{Q}}_8(\mathbb{I}_{\{x_8\}}|X_7) \odot [g(X_7) - \mu]$$

Basics of the algorithm: summary

$$
\begin{array}{ccccc}
 & & ? & & \\
 & & \hat{} & & \\
X_5 & \rightarrow & X_7 & \rightarrow & X_9 \\
\downarrow & & \downarrow & & \\
X_6 & & X_8 & & \\
\hat{x}_6 & & \hat{x}_8 & &
\end{array}
$$

In summary, we have that

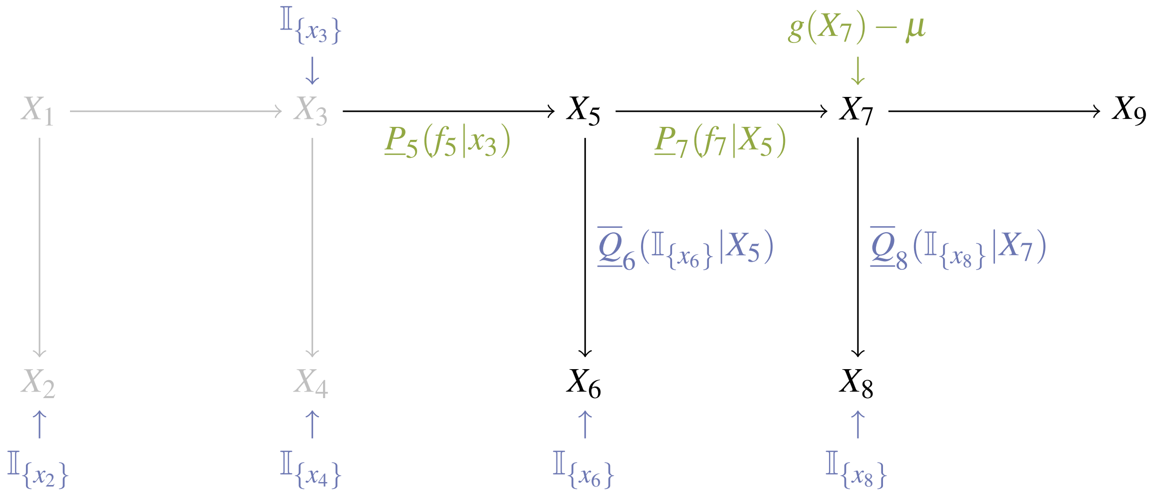
$$\underline{P}_1(f_1) \geq 0 \Leftrightarrow \underline{P}_5(f_5|x_3) \geq 0$$

where

$$\underline{P}_5(f_5|x_3) = \underline{Q}_5(\underline{Q}_6(\{x_6\}|X_5) \odot \underline{P}_7(f_7|X_5)|x_3)$$

$$\underline{P}_7(f_7|X_5) = \underline{Q}_7(\overline{\underline{Q}}_8(\mathbb{I}_{\{x_8\}}|X_7) \odot [g(X_7) - \mu]|X_5)$$

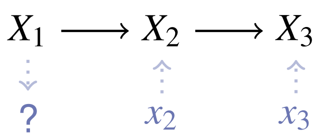
Basics of the algorithm: message passing



$$\underline{P}_7(f_7|X_5) = \underline{Q}_7(\overline{Q}_8(\mathbb{I}_{\{x_8\}}|X_7) \odot [g(X_7)-\mu]|X_5)$$

$$\underline{P}_5(f_5|x_3) = \underline{Q}_5(\underline{Q}_6(\{x_6\}|X_5) \odot \underline{P}_7(f_7|X_5)|x_3)$$

Exercise: MePiCTIr



Let $\mathscr{X}_1 = \{a,b\}$ and $\underline{Q}_1$ be a linear model Q_1 with mass function p_1.

$$Q_1(f) = f(a)p_1(a) + f(b)p_1(b)$$

We also assume that $\underline{Q}_2(\cdot|X_1)$ is a linear model $Q_2(\cdot|X_1)$ with conditional mass function $p(\cdot|X_1)$.

We make no restrictions on the local model $\underline{Q}_3(\cdot|X_2)$ and let $\underline{q} := \underline{Q}_3(\{x_2\}|x_2)$ and $\overline{q} := \overline{Q}_3(\{x_2\}|x_2)$.

Show that (and what happens if $\underline{q} = \overline{q}$?):

$$\underline{R}(\{a\}|x_2,x_3) = \frac{p_1(a)p(x_2|a)\underline{q}}{p_1(a)p(x_2|a)\underline{q} + p_1(b)p(x_2|b)\overline{q}}$$

$$\overline{R}(\{a\}|x_2,x_3) = \frac{p_1(a)p(x_2|a)\overline{q}}{p_1(a)p(x_2|a)\overline{q} + p_1(b)p(x_2|b)\underline{q}}.$$

Exercise: solution

$$f := \mathbb{I}_{\{x_2\}}(X_2)\mathbb{I}_{\{x_3\}}(X_3)[I_{\{a\}}(X_1)-\mu] \text{ with } \mu \in [0,1]$$

We want to find $\underline{P}_{\{1,2,3\}}(f)$ where the joint $\underline{P}_{\{1,2,3\}}$ is given by the recursion equations:

$$\begin{aligned}
\underline{P}_3(\cdot|X_2) &:= \underline{Q}_3(\cdot|X_2)\\
\underline{P}_{\{2,3\}}(\cdot|X_1) &:= Q_2(\underline{P}_3(\cdot|X_2)|X_1) = Q_2(\underline{Q}_3(\cdot|X_2)|X_1)\\
\underline{P}_{\{1,2,3\}} &:= Q_1(\underline{P}_{\{2,3\}}(\cdot|X_1)) = Q_1(Q_2(\underline{Q}_3(\cdot|X_2)|X_1))
\end{aligned}$$

So

$$\begin{aligned}
g(X_1,X_2) &:= \underline{Q}_3(\mathbb{I}_{\{x_2\}}(X_2)\mathbb{I}_{\{x_3\}}(X_3)[I_{\{a\}}(X_1)-\mu]|X_2)\\
&= \mathbb{I}_{\{x_2\}}(X_2)\underline{Q}_3(\mathbb{I}_{\{x_3\}}(X_3)[I_{\{a\}}(X_1)-\mu]|x_2)\\
&= \mathbb{I}_{\{x_2\}}(X_2)\,\overline{\underline{q}}\odot[I_{\{a\}}(X_1)-\mu]\\
h(X_1) &:= Q_2(g(X_1,X_2)|X_1) = \begin{cases} \underline{q}\,[1-\mu]p(x_2|a) & \text{if } X_1 = a\\ \overline{q}\,[-\mu]p(x_2|b) & \text{if } X_1 = b\end{cases}
\end{aligned}$$

Exercise: solution

and therefore finally:

$$\begin{aligned}
\underline{P}_{\{1,2,3\}}(f) = Q_1(h(X_1)) &= p_1(a)\underline{q}\,[1-\mu]p(x_2|a) + p_1(b)\overline{q}\,[-\mu]p(x_2|b)\\
&= p_1(a)\underline{q}p(x_2|a) - \mu\left[p_1(a)p(x_2|a) + p_1(b)\overline{q}p(x_2|b)\right]
\end{aligned}$$

leading to

$$\begin{aligned}
\underline{R}(\{a\}\,|x_2,x_3) &= \max\left\{\mu\in\mathbb{R}\colon \underline{P}_{\{1,2,3\}}(f)\geq 0\right\}\\
&= \frac{p_1(a)p(x_2|a)\underline{q}}{p_1(a)p(x_2|a)\underline{q} + p_1(b)p(x_2|b)\overline{q}}
\end{aligned}$$

and similarly for $\overline{R}(\{a\}\,|x_2,x_3)$.

If $\underline{q} = \overline{q}$ then (use Bayes's Theorem):

$$\underline{R}(\{a\}\,|x_2,x_3) = \overline{R}(\{a\}\,|x_2,x_3) = \frac{p_1(a)p(x_2|a)}{p_1(a)p(x_2|a)+p_1(b)p(x_2|b)} = p(a|x_2).$$

What would we like to achieve and convey?

CREDAL TREES UNDER IRRELEVANCE	MePiCTIr	EstiHMM
interpretation of the graphical structure	essential ideas	essential ideas
recursive construction of the joint	example	example

EstiHMM algorithm

See separate set of slides

References

Useful references

- Jasper De Bock and Gert de Cooman.
  State sequence prediction in imprecise hidden Markov models.
  In Frank Coolen, Gert de Cooman, Thomas Fetz, and Michael Oberguggenberger, editors, *ISIPTA'11: Proceedings of the Seventh International Symposium on Imprecise Probability: Theories and Applications*, pages 159–168, Innsbruck, 2011. SIPTA.
- Gert de Cooman, Filip Hermans, Alessandro Antonucci, and Marco Zaffalon.
  Epistemic irrelevance in credal nets: the case of imprecise Markov trees.
  International Journal of Approximate Reasoning, 51:1029–1052, 2010.
- Gert de Cooman, Enrique Miranda, and Marco Zaffalon.
  Independent natural extension.
  Artificial Intelligence, 175:1911–1950, 2011.