COHERENT UPPER CONDITIONAL PREVISIONS AND THEIR CHOQUET INTEGRAL REPRESENTATION WITH RESPECT TO HAUSDORFF OUTER MEASURES

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OVERVIEW

In a metric space

- **A new model of coherent upper conditional prevision based on Hausdorff outer measures is introduced.**

If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension

- **The given coherent upper conditional prevision is proven to be monotone, comonotonically additive, submodular and continuous from below.**

If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension

- **A coherent upper conditional prevision is characterized as the Choquet integral with respect to the upper conditional probability defined by its associated Hausdorff outer measure.**

This presentation consists of three parts:

- Separately coherent upper conditional previsions defined by Hausdorff outer measures
- Hausdorff outer measures
- Choquet integral representation

Separately coherent upper conditional previsions defined by Hausdorff outer measures

Separately coherent upper conditional previsions

Motivations

The model

Main results

Applications

Separately coherent upper conditional previsions

Metric spaces

Let Ω be a non-empty set and $d: \Omega \times \Omega \rightarrow \mathbb{R}$ a function such that

1M) $\forall\, x,\ y \in \Omega \qquad d(x, y) = 0 \iff x = y$

2M) $\forall\, x, y, z \in\ \Omega \quad d(x, y) + d(x, z) \geq d(y, z)$

d is called a metric on Ω and the pair (Ω, d) is a metric space.

If in 2M) $y = z$, from 1M) we have

$$d(x, y) \geq 0$$

If in 2M) $x = z$ we have $\quad d(x, y) \geq d(y, x)$.

Since it holds for every $x, y \in \Omega$ we have $\quad d(y, x) \geq d(x, y)$

That is $$d(x, y) = d(y, x).$$

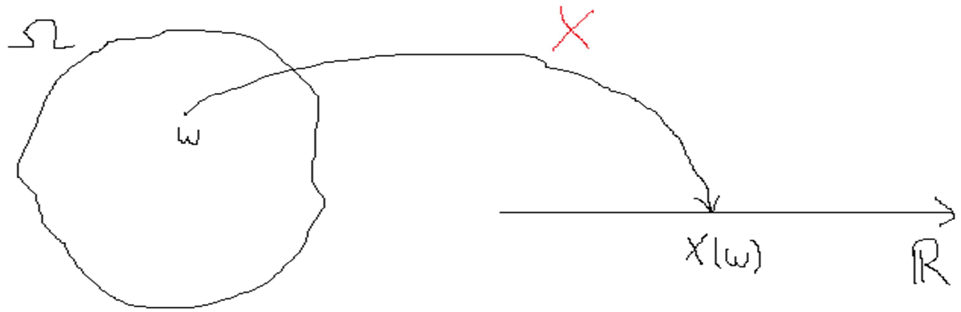
A topology, called the *metric topology*, can be introduced in any metric space by defining the open sets of the space as the sets *G* with the property:

If x is a point of G, then for some r > 0 all points y with $d(x, y) < r$ also belong to G.

The *Borel sigma-field* is the sigma-field generated by the open sets. The Borel sets include the closed sets (as complement of the open sets), the F_σ-sets (contable unions of closed sets) and the G_σ-sets (countable intersection of open sets), etc.

Let (Ω, d) be a metric space and let ***B*** be a Borel partition of Ω, i.e. all sets of the partition are Borel sets.

A random variable X is a bounded function from Ω to the real numbers $\mathbb{R}$. This function is called *gamble* in Walley (1991) or *random quantity* in de Finetti (1972, 1974).



No measurability condition is required for X since a coherent probability can be defined on an arbitrary domain.

In the sequel when a measurability condition for a random variable is required, for example to define the Choquet integral, it is explicitly mentioned.

Remark In the axiomatic approach probability is defined on the sigma-field of the underlying probability space then random variables are required to be functions *measurable with respect to this sigma-field* so that the probability of the inverse image of every Borel set of real numbers is defined.

Let **G** be a sigma field of subsets of Ω. A random variable X is **G**-*measurable* or *measurable with respect to the sigma-field* **G** if the inverse image of every Borel set C of $\mathbb{R}$ belongs to **G.**

$$X^{-1}(C) = \{\omega \in \Omega : X(\omega) \in C\} \in \mathbf{G}$$

For B in ***B*** let $X|B$ be the restriction to B of a random variable X defined on Ω and let sup $X|B$ be the supreme value assumed by X on B.

Let **K**(B) be a linear space of bounded random variables on B. When all bounded random variables on B are considered the linear space is denoted by **L**(B).

For B in ***B*** and $X|B$ in **K**(B) a coherent upper conditional prevision $\overline{P}(X|B)$ is a real functional on **K**(B) such that the following conditions hold for every $X|B$ and $Y|B$ in **K**(B) and positive constant λ:

1. $\overline{P}(X|B) \leq \sup X|B$

2. $\overline{P}(\lambda X|B) = \lambda \overline{P}(X|B)$ positive homogeneity

3. $\overline{P}(X + Y|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ subadditivity

4. $\overline{P}(B|B) = 1$

If $\overline{P}(X|B)$ is a coherent upper conditional prevision on a linear space **K**(B) then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$.

If for every $X|B$ belonging to **K**(B) we have

$$P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$$

then $P(X|B)$ is called a coherent linear conditional prevision and it is a linear, positive functional on **K**(B).

A class of bounded random variables is called a *lattice* if it is closed under point-wise maximum $\vee$ and point-wise minimum $\wedge$.

Two random variables X and Y are *comonotonic* on B if

$$\big(X(\omega_1) - X(\omega_2)\big) \times (Y(\omega_1) - Y(\omega_2)) \geq 0 \qquad \forall\, \omega_1, \omega_2 \in B$$

Let $\overline{P}(X|B)$ be a coherent upper conditional prevision defined on a linear lattice **K**(B) of bounded random variables on *B*. For every $X|B$, $Y|B$, $X_n|B$ in **K**(B) $\overline{P}(X|B)$ is

(i) *monotone* iff $X|B \leq Y|B$ implies $\overline{P}(X|B) \leq \overline{P}(Y|B)$

(ii) *comonotonically additive* iff

$$\overline{P}(X+Y|B) = \overline{P}(X|B) + \overline{P}(Y|B)$$

if X and Y are comonotonic on B.

(iii) *submodular* iff

$$\overline{P}(X \vee Y|B) + \overline{P}(X \wedge Y|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$$

(iv) *continuous from below* iff

$$\lim_{n\to\infty} \overline{P}(X_n|B) = \overline{P}\,(X|B)$$

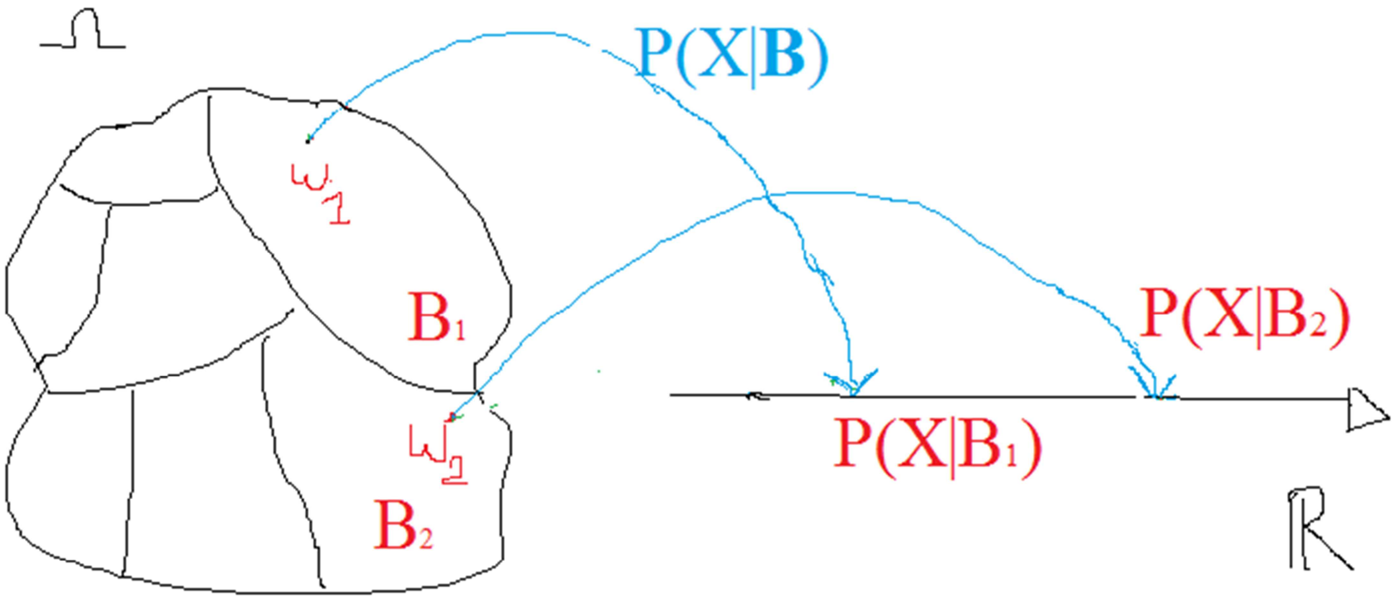
if X_n is an increasing sequence of random variables converging to X.

If for each B in **B** and $X|B$ in **K**(B) coherent upper conditional previsions $\overline{P}(X|B)$ satisfy condition 1, 2, 3, 4 they are called separately coherent upper conditional previsions and can be extended to a common domain **H** containing all constants (Walley, 1991, p.291) .

For each X in **H** let $\overline{P}(X|\boldsymbol{B})$ be the function defined on Ω by

$$\overline{P}(X|\boldsymbol{B})(\omega) = \overline{P}(X|B) \qquad \text{if } \omega \in B.$$

$\overline{P}(X|\boldsymbol{B})$ is called a *coherent upper conditional prevision* and it is separately coherent if all the $\overline{P}(X|B)$ are separately coherent.


Ω
P(X|B)
ω1
B1
P(X|B2)
ω2
P(X|B1)
B2
R

A random variable X on Ω is **B-**measurable or measurable with respect to the partition **B** if it is constant on the atoms of the partition **B** (Walley 1991, p.291).

Necessary condition for coherence

If for every $B \in \boldsymbol{B}$ $P(X|B)$ are coherent linear conditional previsions and X is ***B***-measurable then (Walley 1991, p. 292)

$$P(X|\boldsymbol{B}) = X$$

Motivations

Why the necessity to propose a new model of separately coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$?

Because separately coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ cannot always be defined as extension of conditional expectation $E(X|\boldsymbol{G})$ of measurable random variables.

In fact conditional expectation $E(X|\boldsymbol{G})$ defined by the Radon-Nikodym derivative, according to the axiomatic definition, may fail to be separately coherent.

It occurs because one of the defining properties of the Radon- Nikodym derivative, that is to be measurable with respect to the sigma-field of the conditioning events, contradicts a necessary condition for the coherence ($P(X|\boldsymbol{B}) = X$).

Linear separately coherent conditional prevision $P(X|\boldsymbol{B})$ can be compared with conditional expectation $E(X|\boldsymbol{G})$ if the partition ***B*** generates the sigma-field ***G*** (Koch, 1997, p. 262)

$$E(X|\boldsymbol{G})(\omega) = P(X|B) \quad \text{if } \omega \in B \quad \text{with } B \in \boldsymbol{B}.$$

Definition of conditional expectation (Billingsley, 1986)

Let $\mathbf{F}$ and $\mathbf{G}$ be two sigma-fields of subsets of $\mathbf{\Omega}$ with $\mathbf{G} \subset \mathbf{F}$ and let X be an integrable, $\mathbf{F}$-measurable random variable. Let P be a probability measure on $\mathbf{F}$; define a measure ν on $\mathbf{G}$ by

$$\nu(G) = \int_G XdP.$$

This measure is finite and absolutely continuous with respect to P (i.e. $P(A) = 0 \implies \nu(A) = 0 \quad \forall A \in \mathbf{G}$).

Thus there exists a function, the Radon-Nikodym derivative, denoted by $E(X|\boldsymbol{G})$, $\boldsymbol{G}$-measurable, integrable and satisfying the functional equation

$$\int_G E(X|\boldsymbol{G})dP = \int_G XdP \qquad \text{with } G \text{ in } \boldsymbol{G}.$$

This function is unique up to a set of P-measure zero and it is a version of the conditional expectation value.

If linear conditional prevision $P(X|\boldsymbol{B})$ is defined by the Radon-Nikodym derivative the necessary condition for coherence $P(X|\boldsymbol{B}) = X$ is not always satisfied.

Theorem 1 Let $\Omega = [0,1]$, let $\mathbf{F}$ be the Borel sigma-field of [0,1] and let P be the Lebesgue measure on $\mathbf{F}$. Let $\mathbf{G}$ be a sub sigma-field properly contained in $\mathbf{F}$ and containing all singletons of [0,1]. Let ***B*** be the partition of all singletons of [0,1] and let X be the indicator function of an event *A* belonging to $\mathbf{F}$-$\mathbf{G}$. If we define the linear conditional prevision

$P(X|\{\omega\})$ equal to the Radon-Nikodym derivative with probability 1, that is

$$P(X|\{\omega\}) = E(X|\boldsymbol{G})$$

except on a set *N* of [0,1] of P-measure zero, then the conditional prevision $P(X|\{\omega\})$ is not coherent.

(Theorem 1 S. Doria, Annals of Operation Research, Vol. 195, pp. 38-44, 2012)

If the equality $P(X|\{\omega\}) = E(X|\boldsymbol{G})$ holds with probability 1, the linear conditional prevision $P(X|\{\omega\})$ is different from X, the indicator function of A. In fact having fixed A in $\mathbf{F}$-$\mathbf{G}$ the indicator function of A is not $\mathbf{G}$-measurable.

It occurs because for every Borel set C

$$X^{-1}(C) = \{\omega \in \Omega : X(\omega) \in C\} =$$

$$= \begin{cases} \emptyset & \text{if } 0, 1 \notin C \\ A & \text{if } 1 \in C \text{ and } 0 \notin C \\ A^c & \text{if } 0 \in C \text{ and } 1 \notin C \\ \Omega & \text{if } 0, 1 \in C \end{cases}$$

and since $A \notin \mathbf{G}$ then X is not $\mathbf{G}$-measurable.

Example 1 (Billingsley, 1986; Seidenfeld et al. 2001)

Let Ω = [0,1]

F = Borel sigma-field of Ω,

P = the Lebesgue measure on **F**

G = the sub sigma-field of sets that are either countable or co-countable

B = the partition of all singletons of [0,1].

If the linear conditional prevision is equal, with probability 1, to conditional expectation defined by the Radon-Nikodym derivative, we have that

$$P(X|\boldsymbol{B}) = E(X|\boldsymbol{G}) = P(X)$$

since the events in $\boldsymbol{G}$ have probability either 0 or 1.

So when X is the indicator function of an event A = [a,b] with 0 < a < b < 1 then

$$\overline{P}(X|\mathbf{B}) = \overline{P}(A)$$

and it does not satisfy the necessary condition for coherence, that is

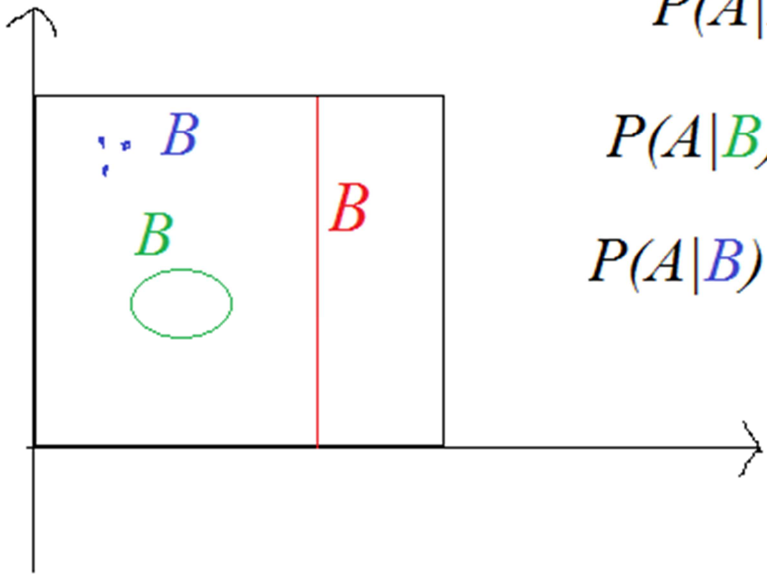
$$\overline{P}(X|\{\omega\}) = X .$$

Evident from Theorem 1 and Example 1 is the necessity to introduce a new mathematical tool to define coherent conditional previsions.

The model

For every $B \in \boldsymbol{B}$ denote by *s* the Hausdorff dimension of the conditioning event *B* and by h^s the Hausdorff s-dimensional outer measure.

h^s is called the Hausdorff outer measure *associated* with the coherent upper conditional prevision $\overline{P}(X|B)$.



$$P(A|B)=h_1(AB)|h_1(B)$$

$$P(A|B)=h_2(AB)|h_2(B)$$

$$P(A|B)=h_0(AB)|h_0(B)$$

Theorem 2. Let **L**(B) be the class of all bounded random variables on B and let m be a 0-1-valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the functional $\overline{P}(X|B)$ defined on **L**(B) by

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \qquad \text{if } 0 < h^s(B) < +\infty$$

$$\overline{P}(X|B) = m(XB) \qquad \text{if } h^s(B) = 0, +\infty$$

is a coherent upper conditional prevision.

(Theorem 2 , S. Doria, Annals of Operation Research, Vol. 195, pp.38-44, 2012)

The unconditional coherent upper prevision $\overline{P}(X|\Omega)$ is obtained as a particular case when the conditioning event is Ω.

Coherent upper conditional probabilities $\overline{P}(A|B)$ are obtained when only 0-1 valued random variables are considered.

Theorem 3 Let *m* be a 0-1-valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different *m* is chosen for each *B*. Then for each $B \in \boldsymbol{B}$ the function $\overline{P}(X|B)$ defined on $\wp(B)$ by

$$\overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)} \qquad if \quad 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(A|B) = m(AB) \qquad if \quad h^s(B) = 0, +\infty$$

is a coherent upper conditional probability.

Main Results

Let B be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension.

Denote by

$$\mu_B^*(A) = \overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)}$$

the coherent upper conditional probability defined on $\wp(B)$.

From Theorem 2 we have that the coherent upper conditional prevision $\overline{P}(\cdot\,|B)$ is a functional on $\mathbf{L}(B)$ with values in $\mathbb{R}$ and the coherent upper conditional probability μ_B^* integral represents $\overline{P}(X|B)$ since

$$\overline{P}(X|B) = \int X d\mu_B^* = \frac{1}{h^s(B)} \int_B X dh^s$$

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-Separately coherent upper conditional previsions
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If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension and **K**(B) is a linear lattice of bounded random variables containing all constants

Necessary and sufficient conditions for a coherent upper conditional prevision to be uniquely represented as the Choquet integral with respest to the upper conditional probability defined by its associated Hausdorff outer measure are to be :

Monotone

Comonotonically additive

Submodular

Continuous from below

A necessary and sufficient condition for an upper prevision $\overline{P}$ to be coherent is to be the *upper envelope* of linear previsions, i.e. there exists a class M of linear previsions, defined on a same domain, such that

$$\overline{P}(X) = \sup\{P(X): P \in M\}$$

Given a coherent upper prevision $\overline{P}$ defined on a domain **K,** the maximal coherent extension of $\overline{P}$ to the class of all bounded random variables is called *natural extension* of $\overline{P}$ (Walley, 1991, 3.1.1)

The linear extension theorem (Walley, 1991, 3.4.2) assures that the class of all linear extensions to the class of all bounded random variables of a linear prevision P defined on a linear space **K** is the class $M(P)$ of all linear previsions that are dominated by P on $\mathbf{K}$.

Moreover the upper and lower envelopes of $M(P)$ are the natural extensions of P (Walley, 1991, 3.4.3).

For every conditioning event B let $P(\cdot\,|B)$ be the restriction to the class of all bounded Borel-measurable random variables on B of the coherent upper conditional prevision $\overline{P}(\cdot\,|B)$ defined as in Theorem 2.

$\overline{P}(\cdot\,|B)$ is the upper envelope of all linear extensions of $P(\cdot\,|B)$ to the class of all bounded random variables on B

If the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension, the previous result is due to the fact $P(\cdot|B)$ is a linear prevision and its natural extension to the class of all bounded random variables is given by

$$\overline{P}(X|B) = \int_B X d\mu_B^* = \sup_{\alpha \in M} \int_B X d\alpha$$

since Hausdorff outer measures are **submodular**.

(Denneberg, 1994, Proposition 10.3)

Let μ be a monotone set function on an algebra $\mathcal{A} \subset \wp(\Omega)$ and define

$$M = \left\{ \begin{matrix} \alpha |\ \alpha \text{ additive on } \mathcal{A}, \alpha(\Omega) = \mu(\Omega), \\ \alpha \leq \mu \end{matrix} \right\}$$

μ is submodular if and only if $M \neq \emptyset$ and

$$\int_B Xd\mu = \sup_{\alpha\in M} \int_B Xd\alpha.$$

Under this condition μ is the upper envelope of M ,

$\mu = \sup_{\alpha\in M}\alpha.$

For finite μ the set M writes as

$$M = \left\{\alpha \middle| \ \alpha \text{ additive on } \mathcal{A}\ \overline{\mu} \leq \alpha \leq \mu \right\}$$

$\overline{\mu}(A) = \mu(\Omega) - \mu(A^c)$ and $\overline{(\mu^*)} = (\overline{\mu})_*$

If $\mathcal{A}$ is the sigma-field of Borel and $\mu = \mu_B^*$

M is a singleton containing the countably additive Hausdorff s-dimensional measure.

Applications

If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension the given model of coherent upper conditional prevision can be used to analyze complex situations where the single subjects have no influence on the situation; aggregative behaviors of "large" sets can change the final result.

Market premium functional

- Insurance prices are determined by the collettive efforts off all agents (buyers and sellers).
- Individual insurers are not price-makers, the insurance price cannot be influenced by the action of a single insurer.
- Wang et al., Axiomatic characterization of insurence prices, Insurance: Mathematics and Economics, 21, 173-183, 1997.

Game theory

- Non-atomic games with infinetly many players.
- Single players have no influence on the the situation. Aggregative behaviors can cange the the payoffs.
- Example can be elections, traffic simulation.
- Aumann, Shapley, Markets with a continuum of players, 32, 93-50, 1964.

Cahotic systems

- The model of coherent upper conditional prevision can be used to make prevision in cahotic systems.
- The final state of these systems is often represented by a strange attractor that has fractional Hausdorff dimension.

Coherent risk measures

- A risk measure satisfying the four axioms of translation invariance, subadditivity, positive homogeneity and monotonicity is called coherent.
- The functional $P(X|B)$ satisfies these axioms.
- Artzner et al., Coherent Measures of Risk, Math. Finance, 3, 203-228, 1999

Connections with other uncertainty measures

Capacities and Hausdorff measures

Dellacherie C., Ensembles analytiques, capacities, measures de Hausdorff, Springer, 1972

Hausdorff outer measures

Given a non-empty set Ω, let $\wp(\Omega)$ be the class of all subsets of Ω. An *outer measure* is a function $\mu^*\colon \wp(\Omega) \rightarrow [0, +\infty]$

such that

$\mu^*(\emptyset) = 0,$

$\mu^*(A) \leq \mu^*(A')$ if $A \subseteq A'$ (monotonicity)

$\mu^*(\bigcup_{i=1}^{+\infty} A_i) \leq \sum_{i=1}^{+\infty} \mu^*(A_i)$ (subadditivity)

Given an additive set function μ defined on a class S properly contained in the power-set $\wp(\Omega)$ its outer measure μ^* and inner measure μ_* are defined on $\wp(\Omega)$ by

$$\mu^*(A) = \inf\{\mu(B): B \supset A; B \in S\}, \quad A \in \wp(\Omega)$$

$$\mu_*(A) = \sup\{\mu(B): B \subset A; B \in S\}, \quad A \in \wp(\Omega)$$

Examples of outer measures are the Hausdorff outer measures (named for Felix Hausdorff 1868 – 1942)

(Rogers C., 1970, Falconer K., 1986)

Let (Ω,d) be a metric space.

The *diameter* of a non - empty subset U of Ω is defined as

$$|U| = \sup\{d(x,y) : \ x,y \in U\}$$

For $\delta > 0$ the class $\{U_i\}_{i=1}^{+\infty}$ is called a *δ-cover* of a subset E of Ω if

$$E \subseteq \bigcup_{i=1}^{+\infty} U_i \qquad \text{and} \qquad 0 < |U_i| \leq \delta$$

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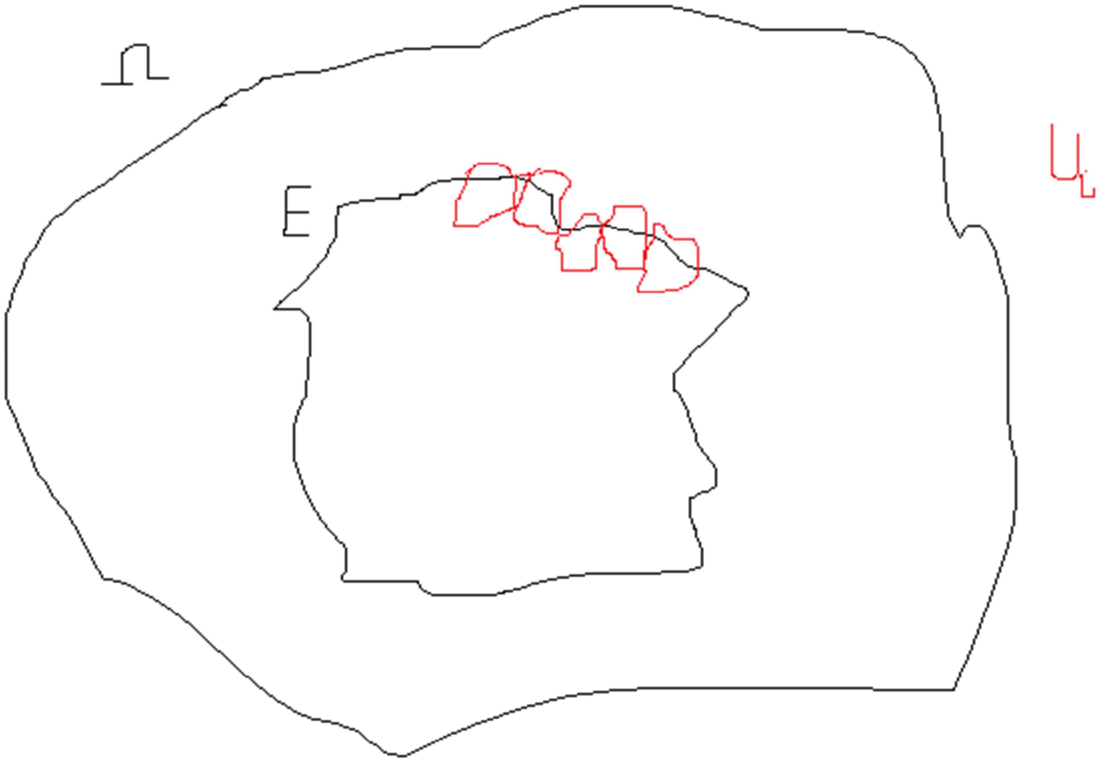


Let s be a non-negative number. We define the function

$$h_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s$$

where the inferior is over all countable δ-covers $\{U_i\}_{i=1}^{+\infty}$.

The *Hausdorff s-dimensional outer measure* of *E*, denoted by h^s, is defined as

$$h^s(E) = \lim_{\delta \mapsto 0} h^s_\delta(E)$$

This limit exists, but may be infinite, since h^s_δ increases as δ decreases because less δ-covers are available.

Remark 1 : The limit of a monotone decreasing function when the variable δ goes to the inferior of its possible values exists and is equal to the supreme value of the possible values of the function.

Remark 2: The function h_δ^s is a decreasing function on δ,

For every δ and δ_1 : $\delta < \delta_1 \Longrightarrow h_\delta^s(E) \geq h_{\delta_1}^s(E)$

In fact if $\delta < \delta_1$ the class of δ-covers is less than the class of δ_1-covers because all the γ-covers with $\delta < \gamma < \delta_1$ are δ_1-covers but not δ-covers.

So for every subset *E* of Ω the function h_δ^s is a decreasing function on δ since when δ decreases the inferior increases because it is calculated on a smaller class of δ-covers.

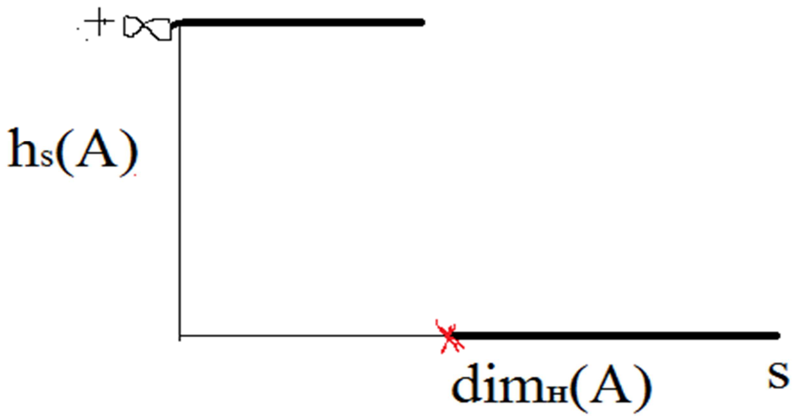
The Hausdorff dimension of a set A , $\dim_{\mathrm{H}}(\mathrm{A})$, is defined as the unique value, such that

$$h^s(A) = +\infty \quad \text{if} \quad 0 \leq s < \dim_{\mathrm{H}}(A)$$

$$h^s(A) = 0 \qquad \text{if} \quad \dim_{\mathrm{H}}(A) < s < +\infty$$

If $0 < h^s(A) < +\infty$ then $\dim_{\mathrm{H}}(A) = s$, but the converse is not true (If A is a countable set → $\dim_{\mathrm{H}}(A) = 0$ and $h^0(A) = +\infty$).

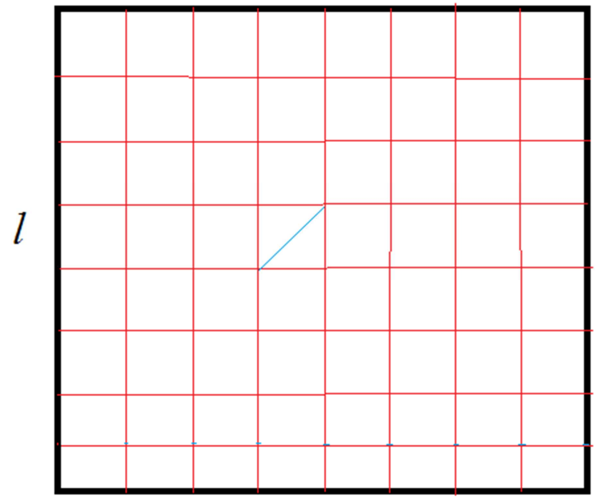
The Hausdorff dimension is the critical value of *s* at which the jump from $+\infty$ to 0 occurs (Falconer 1990).


+∞
hs(A)
dimH(A)
s

Example

Let (Ω, d) be the Euclidean metric space with $\Omega = [0,1]^2$.

Let $E \subseteq \Omega$ be a square of side l.



$$h_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s$$

$$|U_i| = \sqrt{\left(\frac{l}{n}\right)^2 + \left(\frac{l}{n}\right)^2} = \frac{l}{n}\sqrt{2} < \delta$$

$$h_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s = n^2 \left(\frac{l}{n}\sqrt{2}\right)^s$$

$$h^s(E) = \lim_{\delta \mapsto 0} h^s_\delta(E) = \lim_{n\to\infty} n^2 \left(\frac{l}{n}\sqrt{2}\right)^s$$

$s = 0$ $$h^0(E) = \lim_{n\to\infty} n^2 \left(\frac{l}{n}\sqrt{2}\right)^0 = +\infty$$

$s = 1$ $$h^1(E) = \lim_{n\to\infty} n^2 \left(\frac{l}{n}\sqrt{2}\right)^1 = +\infty$$

$s = 2$ $$h^2(E) = \lim_{n\to\infty} n^2 \left(\frac{l}{n}\sqrt{2}\right)^2 = 2l^2$$

$s = 3$ $$h^3(E) = \lim_{n\to\infty} n^2 \left(\frac{l}{n}\sqrt{2}\right)^3 = 0$$

So,

$$0 < \ h^2(E) < +\infty \Longrightarrow \dim_{\mathrm{H}}(E) = 2$$

A subset A of Ω is called *measurable* with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is

$$h^s(E) = h^s(A \cap E) + h^s(E - A)$$

for all sets $E \subseteq \Omega$.

The class of all measurable sets is a sigma-field.

Hausdorff *s*-dimensional outer measures are *metric* outer measures,

$$h^s(E \cup F) = h^s(E) + h^s(F)$$

whenever $d(E,F) = \inf\{d(x,y): x \in E, y \in F\} > 0$

All Borel subsets of Ω are measurable with respect to any metric outer measure (Falconer 1986, Theorem 1.5)

So every Borel subset of Ω is *measurable* with respect to every Hausdorff outer measure h^s since Hausdorff outer measures are metric.

The restriction of the Hausdorff outer measure h^s to the sigma-field of h^s-measurable sets, containing the sigma-field of the Borel sets, is called *Hausdorff s-dimensional measure.*

The Hausdorff 0-dimensional measure is the counting measure;

the Hausdorff 1-dimensional measure is the Lebeasgue measure

The Hausdorff *s*-dimensional measures are *modular* on the Borel sigma-field, that is

$$h^s(A \cup B) + h^s(A \cap B) = h^s(A) + h^s(B)$$

for every pair of Borel sets *A* and *B*.

So (Denneberg, 1994 Proposition 2.4) Hausdorff outer measures are *submodular* (or *2-alternating*)

$$h^s(A \cup B) + h^s(A \cap B) \leq h^s(A) + h^s(B)$$

Denote by **O** the class of all open sets of Ω and by **C** the class of all compact sets of Ω, the restriction of each Hausdorff *s*-dimensional outer measure to the class **H** of all h^s-measurable sets with finite Hausdorff s-dimensional outer measure is *strongly regular* , that is it is regular

outer regular

(a) $h^s(A) = \inf\{h^s(U) | A \subset U, U \in \mathbf{O}\}$ for all $A \in$ **H** *inner regular*

(b) $h^s(A) = \sup\{h^s(C)|C \subset A, C \in \mathbf{C}\}$ for all $A \in$ **H**

With the additional property

(c) $\inf\{h^s(U - A)|A \subset U, U \in \mathbf{O}\} = 0$ for all $A \in$ **H**.

Any Hausdorff s-dimensional outer measure is continuous from below (Falconer 1986, Lemma 1.3), that is for any increasing sequence $\{A_j\}$ of sets

$$\lim_{j\to\infty} h^s(A_j) = h^s\left(\lim_{j\to\infty} A_j\right)$$

Any Hausdorff s-dimensional outer measure is *translation invariant* (Falconer 1986, p.18)

$$h^s(x+E) = h^s(E)$$

where $x + E = \{x + y : y \in E\}$

Choquet integral representation

Given a family L of functions $X : \Omega \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and a functional **Γ** we say that **Γ** can be represented as the Choquet integral with respect to a monotone set function ν defined on $\wp(\Omega)$ if

$$\Gamma(X) = \int X d\nu$$

for every $X \in \mathrm{L}$.

Denote by M_X the *system of upper level sets*

$$M_X = \{\omega \in \Omega : X(\omega) > x\} = \{X > x\}$$

with $x \in \overline{\mathbb{R}}$

and by $G_{\mu,X}(x)$ the *decreasing distribution function* of X with respect to ν

$$G_{\nu,X}(x) = \nu\{\omega \in \Omega : X(\omega) > x\}$$

If $\nu(\Omega) < +\infty$, the *Choquet integral* of X with respect to ν is defined by

$$\int X d\nu = \int_{-\infty}^{0} \big(G_{\nu,X}(x) - \nu(\Omega)\big) dx + \int_{0}^{+\infty} G_{\nu,X}(x) dx$$

The integral is in $\mathbb{R}$, can assume the values $-\infty, +\infty$ or is undefined when the right-hand side is $+\infty - \infty$.

If $X \geq 0$ or $X \leq 0$ the integral always exists. In particular for $X \geq 0$ we have

$$\int X dv = \int_0^{+\infty} G_{v,X}(x) dx$$

If X is bounded and $\nu(\Omega) = 1$

$$\int X d\nu = \int_{\inf X}^{0} \left(G_{\nu,X}(x) - 1\right) dx + \int_{0}^{\sup X} G_{\nu,X}(x) dx =$$

$$= \int_{\inf X}^{\sup X} G_{\nu,X}(x) dx + \inf X$$

Given a functional Γ there is the problem of determining conditions, which assure that Γ can be represented by the Choquet integral with respect to a monotone set function and to determine the interval of monotone set functions which represent Γ.

Let L be a class of random variables such that

(a) $X \geq 0$ for all $X \in \mathrm{L}$ (non-negativity);

(b) $aX, X \wedge a, X \vee a \in \mathrm{L}$ if $X \in \mathrm{L}, a \in \mathbb{R}^{+}$;

(c) $X \wedge Y, X \vee Y \in \mathrm{L}$ if $X, Y \in \mathrm{L}$ (lattice property).

Uniqueness of the representing set function

(Denneberg, 1994, Proposition 13.5)

If a functional Γ, defined on a domain L is monotone, comonotonically additive, submodular and continuous from below then Γ is representable as Choquet integral with respect to a monotone, submodular set function, which is continuous from below.

Furthermore all monotone set functions on $\wp(\Omega)$ with these properties agree on the set system of weak upper level set

$$M = \left\{ \{\omega \in \Omega : X(\omega) \geq x\} : X \in L; x \in \mathbb{R}^{+} \right\}$$

Remark

If Omega is finite continuity from below of the monotone set function is dropped but the upper (conditional) prevision is defined as the Choquet integral with respect to the Hausdorff outer measure of order 0, which is the counting measure. The counting measure is the ONLY measure which is countably additive (and so continuous from below) on the power set of every set Omega (countable or uncountable).

If the monotone set function ν is defined on a class S properly contained in $\wp(\Omega)$ a random variable X is called *upper ν-measurable* if

$$G_{\nu^*,X}(x) = G_{\nu_*,X}(x) \quad \text{e.c.}$$

and the Choquet integral is the common value

$$\int X d\nu = \int X d\nu^* = \int X d\nu_*$$

The finite case

Let Ω be a finite set and let A_i be the atoms of Ω. Let μ be a monotone set function on $\wp(\Omega)$ with $\mu(\Omega) = 1$.

If the A_i are enumerated so that $x_i = X(A_i)$ in descending order, i.e. $x_1 \geq x_2 \geq \ldots x_n$ then

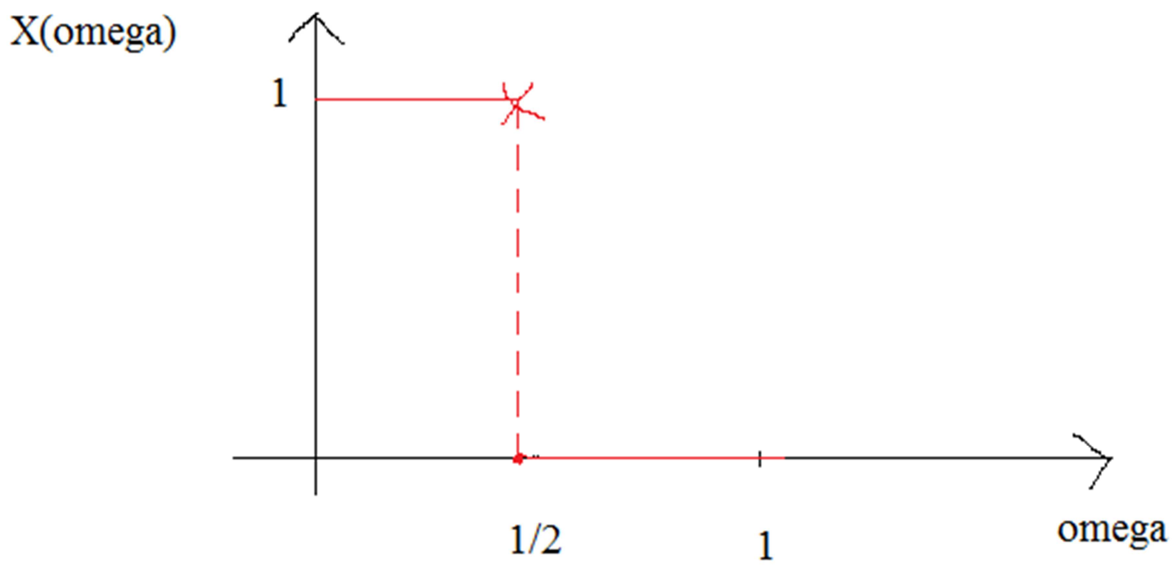
$$\int X d\mu = \sum_{i=1}^{n} (x_i - x_{i+1}) \mu(S_i)$$

where $S_i = A_1 \cup A_2 \cup \ldots \cup A_i \quad i = 1, \ldots, n, \quad x_{n+1} = 0$

Exercise 1

Let $\Omega = [0,1]$ and

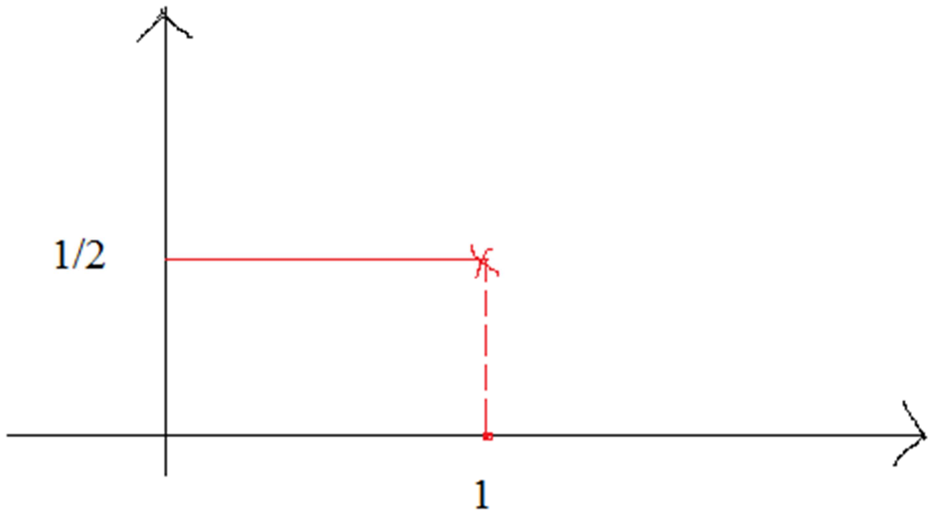
let $X(\omega) = \begin{cases} 1 & if \quad \omega < \frac{1}{2} \\ 0 & if \quad \omega \geq \frac{1}{2} \end{cases}$


X(omega)
1
1/2
1
omega

Determine $P(X|\Omega)$.

$$P(X|\Omega) = \int_0^1 h^1\{\omega \in \Omega : X(\omega) > x\}dx + \inf X =$$

$$= \begin{cases} \int_0^1 \frac{1}{2}dx & if \quad 0 \leq x < 1 \\ 0 & if \quad x = 1 \end{cases}$$


1/2
1

Let $\boldsymbol{B}$ be the partition of singletons of [0,1].

$$P(X|B) = P(X|\{\omega\}) =$$

$$= \int_0^1 h^0\{\omega \in B : X(\omega) > x\}dx + \inf X =$$

$$= \begin{cases} 1 & if \quad \omega < \dfrac{1}{2} \\ 0 & if \quad \omega \geq \dfrac{1}{2} \end{cases}$$

Exercise 2

Let $\Omega = [0,1]$ and

let $X(\omega) = \begin{cases} 1 & if \quad \omega \in \mathbb{Q} \cap [0,1] \\ 0 & if \quad \omega \in \mathbb{R} - \mathbb{Q} \cap [0,1] \end{cases}$

called the Dirichlet function on [0,1].

$$P(X|\Omega) = \int_0^1 h^1\{\omega \in \Omega : X(\omega) > x\}dx + \inf X =$$

$$= h^1\{\mathbb{Q} \cap [0,1]\}$$

That is

$$P(X|\Omega) = 0$$

Let $\boldsymbol{B}$ be the partition of singletons of [0,1].

$$P(X|\{\omega\}) = \int_0^1 h^0\{\omega \in \mathrm{B} : \ X(\omega) > x\}dx + \inf X =$$

$$= \begin{cases} 1 & if \qquad \omega \in \ \mathbb{Q} \cap [0,1] \\ 0 & if \quad \omega \in \mathbb{R} - \mathbb{Q} \cap [0,1] \end{cases}$$

Exercise 3

Let $\Omega = [0,1]$ and let X be the indicator function of a set A, i.e.

$$X(\omega) = \begin{cases} 1 & if \quad \omega \in A \\ 0 & if \quad \omega \notin A \end{cases}$$

Determine $\overline{P}(X|\Omega)$.

$$\overline{P}(X|\Omega) = \int_0^1 h^1\{\omega \in \Omega : X(\omega) > x\}dx + \inf X = h^1(A)$$

Let $\boldsymbol{B}$ be the partition of singletons of [0,1]. Determine $\overline{P}(X|B)$ for every $B \in \boldsymbol{B}$.

$$\overline{P}(X|B) = \overline{P}(X|\{\omega\}) =$$

$$= \int_0^1 h^0\{\omega \in B : X(\omega) > x\}dx + \inf X =$$

$$= \begin{cases} 1 & if \quad \omega \in A \\ 0 & if \quad \omega \notin A \end{cases}$$

Remark

If A is a Lebeasgue measurable set its indicator function is h^1-measurable and the coherent conditional and unconditional previsions are linear.

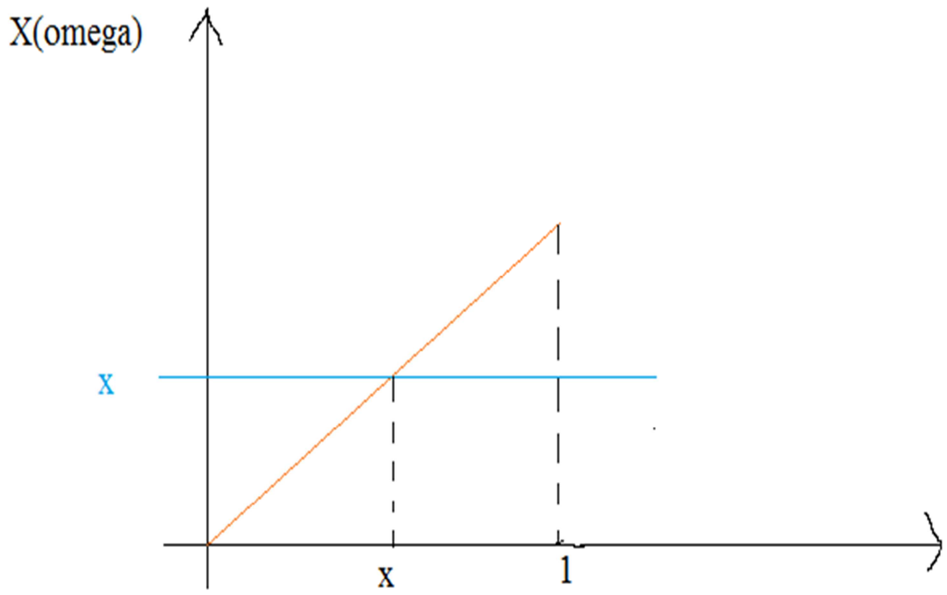
Exercise 4

Let $\Omega = [0,1]$ and let $X(\omega) = \omega$.

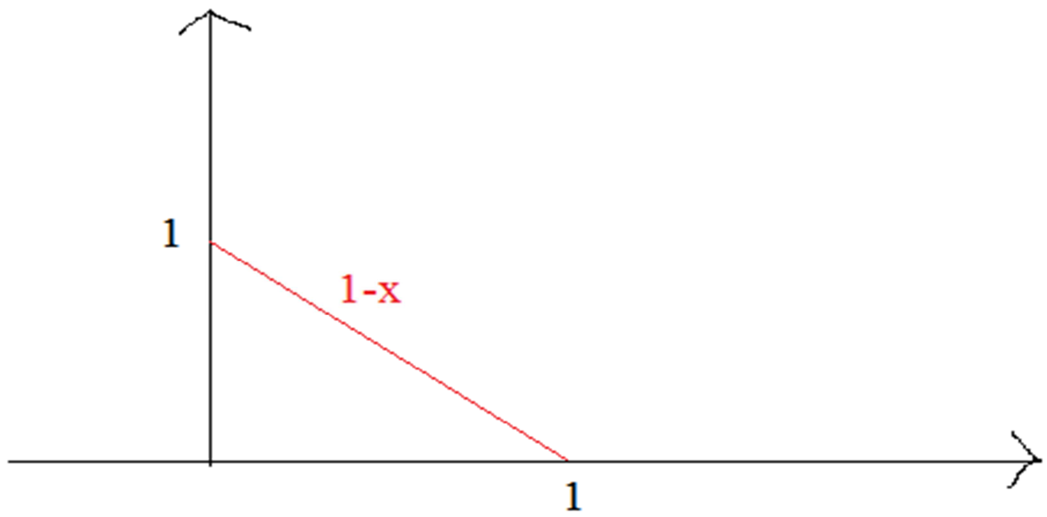
Determine $P(X|\Omega)$.

$$P(X|\Omega) = \int_0^1 h^1\{\omega \in \Omega : \ X(\omega) > x\}dx + \inf X =$$

$$\int_0^1 h^1\{\omega \in \Omega : \ \omega > x\}dx =$$


X(omega)
x
x
1

$$\int_0^1 (1-x)dx = \frac{1}{2}$$


1
1-x
1

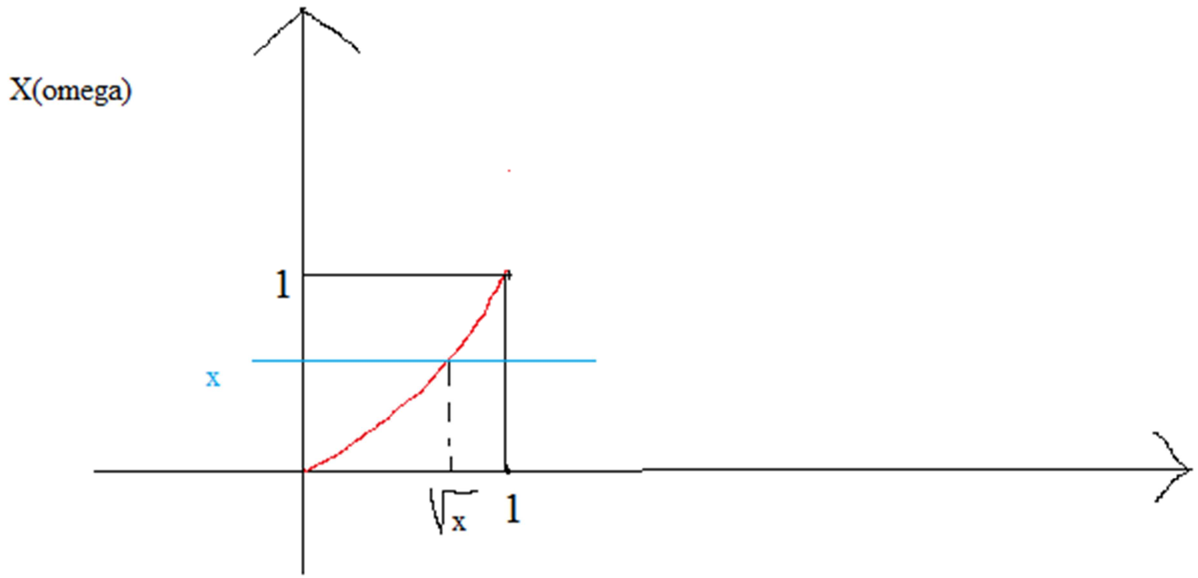
Exercise 5.

Let $\Omega = [0,1]$ and let $X(\omega) = \omega^2$.

Determine $P(X|\Omega)$.

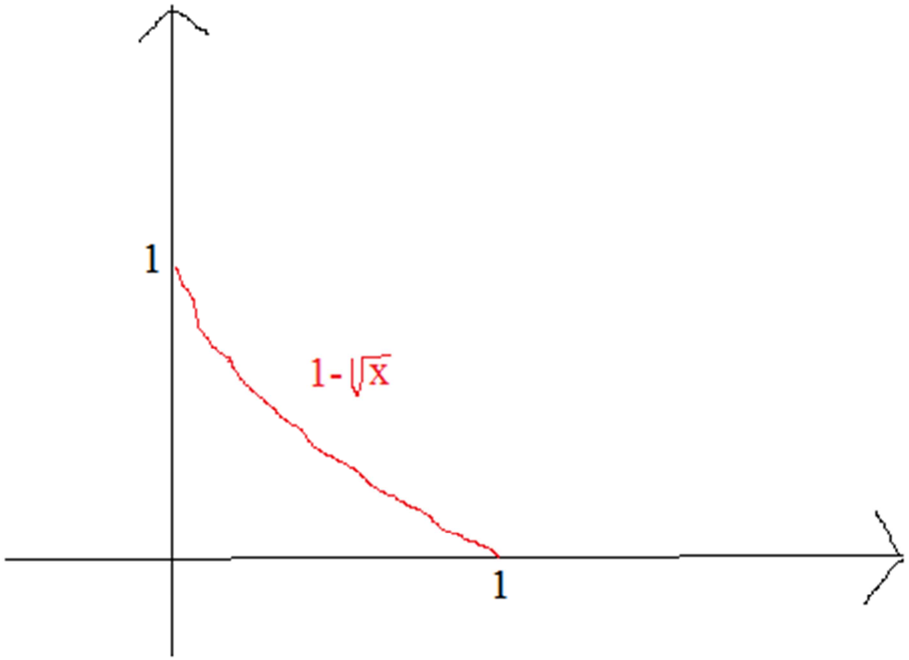
$$P(X|\Omega) = \int_0^1 h^1\{\omega \in \Omega :\ X(\omega) > x\}dx + \inf X =$$

$$\int_0^1 h^1\{\omega \in \Omega : \omega^2 > x\}dx =$$


X(omega)
1
x
$\sqrt{x}$
1

$$\int_0^1 h^1\{\omega \in \Omega : \ \omega > \sqrt{x}\}dx =$$

$$\int_0^1 (1-\sqrt{x})dx = \frac{1}{3}$$


1
1-√x
1

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